

REGULARIZED SOLUTIONS FOR SOME BACKWARD NONLINEAR PARABOLIC EQUATIONS WITH STATISTICAL DATA

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ABSTRACT. In this paper, we study the backward problem of determining initial condition for some class of nonlinear parabolic equations in multidimensional domain where data are given under random noise. This problem is ill-posed, i.e., the solution does not depend continuously on the data. To regularize the instable solution, we develop some new methods to construct some new regularized solution. We also investigate the convergence rate between the regularized solution and the solution of our equations. In particular, we establish results for several equations with constant coefficients and time dependent coefficients. The equations with constant coefficients include heat equation, extended Fisher-Kolmogorov equation, Swift-Hohenberg equation and many others. The equations with time dependent coefficients include Fisher type Logistic equations, Huxley equation, Fitzhugh-Nagumo equation. The methods developed in this paper can also be applied to get approximate solutions to several other equations including 1-D Kuramoto-Sivashinsky equation, 1-D modified Swift-Hohenberg equation, strongly damped wave equation and 1-D Burger's equation with randomly perturbed operator.

1. INTRODUCTION

In this paper, we focus on the problem of finding the initial functions $\mathbf{u}(\mathbf{x}, 0) = u_0(\mathbf{x})$ such that \mathbf{u} satisfies the following nonlinear parabolic equation

$$\begin{cases} \mathbf{u}_t + \mathcal{A}(t, \mathbf{u})\mathbf{u} = F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t), & 0 < t < T, x \in \Omega, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases} \quad (1.1)$$

where the domain $\Omega = (0, \pi)^d$ is a subset of \mathbb{R}^d and $\mathbf{x} := (x_1, \dots, x_d)$. The functions F and G are called the source functions that satisfy the usual Lipschitz and growth conditions. The function H is given and is often called a final value data. The operator \mathcal{A} is given by the Laplacian, or a function of the Laplacian defined by the spectral theorem.

The problem (1.1) is called the backward problem for classical parabolic equation when $\mathcal{A} = -\Delta$. It is applied in fields as the heat conduction theory [5], hydrology [4], groundwater contamination [33], to digitally remove blurred noiseless image [8] and also in many other practical applications of mathematical physics and engineering [37]. It is well-known that the backward parabolic problem is severely ill-posed in the sense of Hadamard [16]. Hence solutions do not always exist, and in the case of existence, the solutions do not depend continuously on the initial data. In fact, from small noise contaminated physical measurements, the corresponding solutions might have large errors. More details on ill-posedness of the problem with random noise can be found in [26].

The analysis of regularization methods for the stable solution (in the sense of Hadamard) of problem (1.1) depends on the mathematical model with the noise term on the source function G and the final value data $\mathbf{u}_T = \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x})$. We suppose that the measurements are described as follows

$$G^{\text{obs}} = G + \text{"noise"}, \quad H^{\text{obs}} = H + \text{"noise"}. \quad (1.2)$$

1.1. Background in the deterministic case. The Problem (1.1) is a generalized form of a class of backward parabolic equations. We give a short history of this problem in the deterministic case. If the "noise" (introduced in (1.2)) is considered as a deterministic quantity, it is natural to study what happens when $\|\text{noise}\|_{L^2} \rightarrow 0$.

When the operator $\mathcal{A}(t) = \mathcal{A}$ (independent of t), regularization results were considered by many authors, we refer the reader to the survey paper of Tuan [38] and the references therein.

When $\mathcal{A}(t)$ depends on t and $F = G = 0$, Lions and Lattes [22] proposed the following quasi-reversibility method:

$$\begin{cases} \mathbf{u}'_\epsilon(t) + \mathcal{A}(t)\mathbf{u}_\epsilon + \epsilon\mathcal{A}^*(t)\mathcal{A}(t)\mathbf{u}_\epsilon &= 0 \\ \mathbf{u}_\epsilon(T) &= \mathbf{u}_T^\epsilon. \end{cases} \quad (1.3)$$

However, Lions and Lattes did not study regularization results for this problem. The regularization result here is still open although some progress has been made. The first paper on this case seems to be that of Krein [12], where he used the log-convexity method to get stability estimates of Hölder type. His method and results have been further developed by Hao and Duc [17].

When $\mathcal{A}(t)$ depends on t and \mathbf{u} , to the best of our knowledge, there do not exist any results on the backward problem. Regularization results in here are very difficult because one can not represent the solution with a nonlinear integral as previously done by others. Hence, the regularized solution can not be obtained with the previous techniques based on nonlinear integral method. Regularization results for problem (1.1) in the deterministic case are still open.

1.2. Background on problem with random noise. If the errors are generated from uncontrollable sources as wind, rain, humidity, etc, then the model is random. If the "noise" (introduced in (1.2)) are modeled as a random quantity, the convergence of estimators $\tilde{\mathbf{u}}(\mathbf{x}, 0)$ of $\mathbf{u}(\mathbf{x}, 0)$ should be studied by statistical methods. Methods applied to the deterministic cases cannot be applied directly for this case. The main idea in using the random noise is of finding suitable estimators $\tilde{\mathbf{u}}(\mathbf{x}, 0)$ and to consider the expected square error $\mathbb{E}[\|\tilde{\mathbf{u}}(\mathbf{x}, 0) - \mathbf{u}(\mathbf{x}, 0)\|^2]$ in a suitable space, also called the mean integrated square error (MISE).

There exist a considerable amount of literature on regularization methods for linear backward problem with random noise. When $F(\mathbf{u}) = 0$, the problem (1.1) is linear and its solution can be defined by a linear operator with random noise

$$u_T = Ku_0 + \text{"noise"}. \quad (1.4)$$

where K is a bounded linear operator that does not have a continuous inverse. There are many well-known methods including spectral cut-off (or called truncation method) of Cavalier [6, 9], the Tikhonov method [11], iterative regularization methods [12]. Mair and Ruymgaart [24] considered theoretical formulae for statistical inverse estimation in Hilbert spaces and applied the method to solve the backward heat problem. Recently, Hohage et al. [20] applied spectral cut-off (truncation method) and Tikhonov-type method to solve linear statistical inverse problems including backward heat equation (See p. 2625, [20]). Recently, Problem (1.1) in the case of $F = 0$ has been studied in [26] in the plane domain.

Until now, to the best of the authors' knowledge, there are only a few results in the case of random source, or random final value observations for nonlinear backward parabolic equation. And there are no results on Problem (1.1). This is our motivation in the present paper. In a few sentences, we give explanation why the nonlinear problem is difficult to investigate. Indeed, when F depends on \mathbf{u} , we can not transform the solution of problem (1.1) into (1.4), this makes the nonlinear problem more challenging. Furthermore, as introduced in the subsection 1.1, if $\mathcal{A} = \mathcal{A}(t, \mathbf{u})$ then we can not transform the problem (1.1) into a nonlinear integral equation, then the previous methods can not be applied for regularizing the problem. **So, our task in this paper is developing and establishing new methods for solving this problem.**

1.3. Outline of the article. In this paper, inspired by the random model in [26], we introduce the following random model in \mathbb{R}^d .

Let $\Omega = (0, \pi)^d \subset \mathbb{R}^d$ for $d \geq 1$. Let us recall the functions H and G from equation (1.1). Let $\mathbf{x}_i = (x_{i_1}, \dots, x_{i_d})$ be grid points of Ω with index $\mathbf{i} = (i_1, i_2, \dots, i_d) \in \mathbb{N}^d$, $1 \leq i_k \leq n_k$ for $k = \overline{1, d}$ where

$$\mathbf{x}_i = (x_{i_1}, \dots, x_{i_d}) = \left(\frac{\pi(2i_1 - 1)}{2n_1}, \frac{\pi(2i_2 - 1)}{2n_2}, \dots, \frac{\pi(2i_d - 1)}{2n_d} \right), \quad i_k = \overline{1, n_k}, \quad k = \overline{1, d}. \quad (1.5)$$

We consider the following nonparametric regression model of data as follows

$$\tilde{D}_{\mathbf{i}} = \tilde{D}_{i_1, i_2, \dots, i_d} := H(x_{i_1}, \dots, x_{i_d}) + \Lambda_{i_1, i_2, \dots, i_d} \Upsilon_{i_1, i_2, \dots, i_d} = H(\mathbf{x}_i) + \Lambda_{\mathbf{i}} \Upsilon_{\mathbf{i}} \quad (1.6)$$

$$\tilde{G}_{\mathbf{i}}(t) = \tilde{G}_{i_1, i_2, \dots, i_d}(t) := G(x_{i_1}, \dots, x_{i_d}, t) + \vartheta \Psi_{i_1, i_2, \dots, i_d}(t) = G(\mathbf{x}_i, t) + \vartheta \Psi_{\mathbf{i}}(t), \quad (1.7)$$

for $i_k = \overline{1, n_k}$, $k = \overline{1, d}$. Here $\Upsilon_{\mathbf{i}} := \Upsilon_{i_1, i_2, \dots, i_d} \sim \mathcal{N}(0, 1)$ and $\Psi_{\mathbf{i}}(t) := \Psi_{i_1, i_2, \dots, i_d}(t)$ are Brownian motions. We assume furthermore that they are mutually independent.

Our main goal in this paper is to provide some regularized solutions that are called estimators for approximating $\mathbf{u}(\mathbf{x}, t)$, $0 \leq t < T$. In this paper, we do not investigate the existence and uniqueness of the solution of backward problem (1.1). The uniqueness of backward parabolic has attracted the attention of many authors; see, for example, [23, 32, 41]. It is also a challenging and open problem, and should be the topic for another paper. In this paper, we assume that the backward problem (1.1) has a unique solution \mathbf{u} (called sought solution) that belongs to an appropriate space. So our main purpose is to consider a regularized problem for finding an approximate solution in the random cases. Furthermore, error estimates with the speed of convergence between the regularized solution and the sought solution under some a priori assumptions on the sought solution are also our primary purpose. In particular, the main purpose in our error estimates is to show that the norm of difference between the regularized solution and the sought solution in $L^2(\Omega)$ tends to zero when $|\mathbf{n}| = \sqrt{n_1^2 + \dots + n_d^2} \rightarrow +\infty$. Our methods in this paper can be applied to solve the backward problem in the deterministic case that was introduced in subsection 1.1.

For the purpose of capturing the main points of the paper, we consider Problem (1.1) and describe our main results for the three cases of $\mathcal{A}(t, \mathbf{u})$.

Case 1: $\mathcal{A}(t, \mathbf{u}) = \mathcal{A}$. In this case, we apply Fourier truncation method associated with knowledge on trigonometric theory in nonparametric regression for establishing regularized solutions. The process for finding the regularized solution is given by the following steps : First, we approximate the given data H and G by approximating functions \widehat{H}_{β_n} and \widehat{G}_{β_n} defined by Theorem (2.1). Then, we express the solution of Problem (1.1) into a nonlinear integral equation which is represented as Fourier series and then we give some regularized solutions which are defined by other nonlinear integral equations. In this case, we will derive rates of convergence under some a priori assumptions of the sought solution \mathbf{u} . Main result in this case is given by Theorems 3.2.

Case 2: $\mathcal{A}(t)$ depends only on t . As discussed before, regularized methods used in Case 1 cannot be applied in this case. Hence, we need to figure out a new regularized method to establish a regularized solution. Our main idea in this case is that of applying a modified Quasi-reversibility method given by Lions[22]. We will not approximate directly the time dependents operator $\mathcal{A}(t)$ as introduced in [22]. Our method is of finding the unbounded time independent operator \mathbf{P} that satisfies conditions in Definition 4.1. Then, we approximate \mathbf{P} by a bounded operator \mathbf{P}_{ρ_n} , in order to establish the well-posedness of the problem associated with the approximating functions \widehat{H}_{β_n} and \widehat{G}_{β_n} . Finding suitable regularized operators is important task in this section. Our main results in this case are Theorem 4.1 and 4.2. A special case of Theorem 4.1 is Corollary 4.1 which studies the **extended Fisher-Kolmogorov equation**.

Case 3: \mathcal{A} depends on t and \mathbf{u} . Using the method in Case 2, we extend the results of Case 2 to the case when the coefficients of \mathcal{A} depends on t and u . Special cases of the equation considered in Theorem 4.2 are **Fisher type Logistic equations** with $F(\mathbf{u}) = a\mathbf{u}(1 - \mathbf{u})$, or **Huxley equation** with $F(\mathbf{u}) = a\mathbf{u}^2(1 - \mathbf{u})$, or **Fitzhugh-Nagumo equation** with $F(\mathbf{u}) = a\mathbf{u}^2(1 - \mathbf{u})(\mathbf{u} - \theta_1)$. See Remark 4.2 for more.

Finally, we want to mention that the backward problem for some concrete nonlocal parabolic equations such as **Ginzburg-Landau equation** where the coefficients of these equations are perturbed by random noises can be studied with the methods in our paper. These equations include the 1-D **Burger's equation**. Furthermore, our analysis and methods in this paper can be applied to get approximate solutions for many well-known equations. We state some examples below. We will work on these problems in a forthcoming paper.

- Backward problem for 1-D **Kuramoto-Sivashinsky equation**:

$$\mathbf{u}_t + d_0 \mathbf{u}_{xx} + (d_1(x) \mathbf{u}_{xx})_{xx} = -\mathbf{u} \mathbf{u}_x + G(x, t), \quad (x, t) \in (0, \pi) \times (0, T), \quad (1.8)$$

with the conditions

$$\begin{cases} \mathbf{u}(0, t) &= \mathbf{u}(\pi, t) = \mathbf{u}_{xx}(0, t) = \mathbf{u}_{xx}(\pi, t) = 0, & t \in (0, T), \\ \mathbf{u}(x, T) &= H(x), & x \in (0, \pi), \end{cases} \quad (1.9)$$

where d_0, d_1 are diffusion coefficients. This nonlinear partial differential equation describes incipient instabilities in a variety of physical and chemical systems (see [3, 10]).

- Backward problem for 1-D **modified Swift-Hohenberg equation**:

$$\mathbf{u}_t + 2\mathbf{u}_{xx} + k\mathbf{u}_{xxxx} + a\mathbf{u} + \mathbf{u}^3 + |\mathbf{u}_x|^2 + G(x, t) = 0, \quad (x, t) \in (0, \pi) \times (0, T), \quad (1.10)$$

with condition (1.9). The Swift-Hohenberg equation is one of the universal equations used in the description of pattern formation in spatially extended dissipative systems, (see [35]), which arise in the study of convective hydrodynamics [36], plasma confinement in toroidal devices [31], viscous film flow and bifurcating solutions of the Navier-Stokes [34].

- Backward problem for **strongly damped wave equation**:

$$\begin{cases} \mathbf{u}_{tt}(\mathbf{x}, t) - \alpha \Delta \mathbf{u}_t(\mathbf{x}, t) - \Delta \mathbf{u}(\mathbf{x}, t) = F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t), & 0 < t < T, \quad \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), & \mathbf{x} \in \Omega \\ \mathbf{u}_t(\mathbf{x}, T) = 0, & \mathbf{x} \in \Omega, \end{cases} \quad (1.11)$$

where $H \in L^2(\Omega)$ is a given function and α is a positive constant. Strongly damped wave equation, occurs in a wide range of applications such as modeling motion of viscoelastic materials [18, 25, 29]. Some more physical applications of the equation (1.11) can be found in [30].

- **1-D Burger's equation**:

$$\begin{cases} \mathbf{u}_t - (A(x, t)\mathbf{u}_x)_x &= \mathbf{u}\mathbf{u}_x + G(x, t), & (x, t) \in \Omega \times (0, T), \\ \mathbf{u}(x, t) &= 0, & x \in \partial\Omega, \\ \mathbf{u}(x, T) &= H(x), & x \in \Omega, \end{cases} \quad (1.12)$$

where $\Omega = (0, \pi)$. The Burgers equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow [2]. The ill-posedness of the backward problem for Burgers equation has been introduced by E. Zuazua et al [18]. The model here is as follows:

Assume the time dependent coefficient $A(x, t)$ is noisy by random data

$$\tilde{A}_k(t) = A(x_k, t) + \bar{\vartheta}\xi_k(t), \quad \text{for } k = \overline{1, n}. \quad (1.13)$$

where $x_k = \frac{(2k-1)\pi}{2n}$, $k = \overline{1, n}$ are the grid points in $(0, \pi)$ and $\xi_k(t)$, $k = \overline{1, n}$ are independent Brownian motions.

2. CONSTRUCTING A FUNCTION FROM DISCRETE RANDOM DATA

In this section, we develop a new theory for constructing a function in $L^2(\Omega)$ from the given discrete random data.

We first introduce notation, and then we state the main results of this paper.

We will occasionally use the following Gronwall's inequality in this paper.

Lemma 2.1. *Let $b : [0, T] \rightarrow \mathbb{R}^+$ be a continuous function and $C, D > 0$ be constants that are independent of t , such that*

$$b(t) \leq C + D \int_t^T b(\tau) d\tau, \quad t > 0.$$

Then we have

$$b(t) \leq Ce^{D(T-t)}.$$

Next we define fractional powers of the Dirichlet Laplacian

$$Af := -\Delta f.$$

Since A is a linear, densely defined self-adjoint and positive definite elliptic operator on the connected bounded domain Ω with Dirichlet boundary condition, using spectral theory, it is easy to show that the eigenvalues of A are given by $\lambda_{\mathbf{p}} = |\mathbf{p}|^2 = p_1^2 + p_2^2 + \cdots + p_d^2$. The corresponding eigenfunctions are denoted respectively by

$$\psi_{\mathbf{p}}(\mathbf{x}) = \left(\sqrt{\frac{2}{\pi}} \right)^d \sin(p_1 x_1) \sin(p_2 x_2) \cdots \sin(p_d x_d). \quad (2.1)$$

Thus the eigenpairs $(\lambda_{\mathbf{p}}, \psi_{\mathbf{p}})$, $p \in \mathbb{N}^d$, satisfy

$$\begin{cases} A\psi_{\mathbf{p}}(\mathbf{x}) = -\lambda_{\mathbf{p}}\psi_{\mathbf{p}}(\mathbf{x}), & \mathbf{x} \in \Omega \\ \psi_{\mathbf{p}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

The functions $\psi_{\mathbf{p}}$ are normalized so that $\{\psi_{\mathbf{p}}\}_{\mathbf{p} \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\Omega)$.

We will use the following notation: $|\mathbf{p}| = |(p_1, \dots, p_d)| = \sqrt{p_1^2 + \dots + p_d^2}$, $|\mathbf{n}| = |(n_1, \dots, n_d)| = \sqrt{n_1^2 + \dots + n_d^2}$.

Definition 2.1. For $\gamma > 0$, we define

$$\mathcal{H}^\gamma(\Omega) := \left\{ h \in L^2(\Omega) : \sum_{p_1=1}^{\infty} \dots \sum_{p_d=1}^{\infty} |\mathbf{p}|^{2\gamma} < h, \psi_{\mathbf{p}} >^2 < \infty \right\}. \quad (2.2)$$

The norm on $\mathcal{H}^\gamma(\Omega)$ is defined by

$$\|h\|_{\mathcal{H}^\gamma(\Omega)}^2 := \sum_{p_1=1}^{\infty} \dots \sum_{p_d=1}^{\infty} |\mathbf{p}|^{2\gamma} < h, \psi_{\mathbf{p}} >^2. \quad (2.3)$$

For any Banach space X , we denote by $L_p(0, T; X)$, the Banach space of measurable real functions $v : (0, T) \rightarrow X$ such that

$$\|v\|_{L^p(0, T; X)} = \left(\int_0^T \|v(\cdot, t)\|_X^p dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|v\|_{L^\infty(0, T; X)} = \text{esssup}_{0 < t < T} \|v(\cdot, t)\|_X < \infty, \quad p = \infty.$$

Let $\beta : \mathbb{N}^d \rightarrow \mathbb{R}$ be a function. Now we state our first main result which gives error estimate between H and $\hat{H}_{\beta_{\mathbf{n}}}$, and error estimate between $\hat{G}_{\beta_{\mathbf{n}}}$ and G .

Theorem 2.1. Define the set $\mathcal{W}_{\beta_{\mathbf{n}}}$ for any $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$

$$\mathcal{W}_{\beta_{\mathbf{n}}} = \left\{ \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d : |\mathbf{p}|^2 = \sum_{k=1}^d p_k^2 \leq \beta_{\mathbf{n}} = \beta(n_1, \dots, n_d) \right\} \quad (2.4)$$

where $\beta_{\mathbf{n}}$ satisfies

$$\lim_{|\mathbf{n}| \rightarrow +\infty} \beta_{\mathbf{n}} = +\infty.$$

For a given \mathbf{n} and $\beta_{\mathbf{n}}$ we define functions that are approximating H, G as follows

$$\hat{H}_{\beta_{\mathbf{n}}}(\mathbf{x}) = \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} \left[\frac{\pi^d}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \tilde{D}_{i_1, i_2, \dots, i_d} \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right] \psi_{\mathbf{p}}(\mathbf{x}) \quad (2.5)$$

and

$$\hat{G}_{\beta_{\mathbf{n}}}(\mathbf{x}, t) = \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} \left[\frac{\pi^d}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \tilde{G}_{i_1, i_2, \dots, i_d}(t) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right] \psi_{\mathbf{p}}(\mathbf{x}). \quad (2.6)$$

Let $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ with $\mu_k > \frac{1}{2}$ for any $k = \overline{1, d}$. Let us choose $\mu_0 \geq d \max(\mu_1, \dots, \mu_d)$. If $H \in \mathcal{H}^{\mu_0}(\Omega)$ and $G \in L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))$ then the following estimates hold

$$\mathbb{E} \left\| \hat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 \leq \overline{\mathcal{C}}(\mu_1, \dots, \mu_d, H) \beta_{\mathbf{n}}^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k} + 4\beta_{\mathbf{n}}^{-\mu_0} \left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2,$$

$$\mathbb{E} \left\| \hat{G}_{\beta_{\mathbf{n}}}(\cdot, t) - G(\cdot, t) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq \overline{\mathcal{C}}(\mu_1, \dots, \mu_d, H) \beta_{\mathbf{n}}^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k} + 4\beta_{\mathbf{n}}^{-\mu_0} \left\| G \right\|_{L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))}^2,$$

where

$$\overline{\mathcal{C}}(\mu_1, \dots, \mu_d, H) = 8\pi^d V_{max}^2 \frac{2\pi^{d/2}}{d\Gamma(d/2)} + \frac{16\mathcal{C}^2(\mu_1, \dots, \mu_d)\pi^{d/2}}{d\Gamma(d/2)} \left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2.$$

Corollary 2.1. *Let H, G be as in Theorem (2.1). Then the term $\mathbf{E} \left\| \hat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \hat{G}_{\beta_{\mathbf{n}}} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2$ is of order*

$$\max \left(\frac{\beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, \beta_{\mathbf{n}}^{-\mu_0} \right).$$

To prove this Theorem, we need some preliminary results.

Lemma 2.2. *Let $\mathbf{p}, \mathbf{q} \in \mathbb{N}^d$ and $\mathbf{p} = (p_1, \dots, p_d)$, $\mathbf{q} = (q_1, \dots, q_d)$ with $p_k = \overline{1, n_k - 1}$ and $x_{i_k} = \frac{\pi(2i_k - 1)}{2n_k}$ for $k = \overline{1, d}$. Then for all $\mathbf{q} \in \mathbb{N}^d$, we have*

$$\begin{aligned} \overline{R}_{\mathbf{p}, \mathbf{q}} &= \frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{q}}(x_{i_1}, \dots, x_{i_d}) \\ &= \begin{cases} \frac{1}{\pi^d}, & p_k \mp q_k = 2l_k n_k, \quad k = \overline{1, d}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (2.7)$$

If $\mathbf{q} \in \mathbb{N}^d$ satisfies that $q_k := \overline{1, n_k - 1}$ then we have

$$\overline{R}_{\mathbf{p}, \mathbf{q}} = \frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{q}}(x_{i_1}, \dots, x_{i_d}) = \begin{cases} \frac{1}{\pi^d}, & p_k = q_k, \quad k = \overline{1, d}, \\ 0, & \exists k = \overline{1, d} \quad p_k \neq q_k, \end{cases} \quad (2.8)$$

Proof. The lemma is a direct consequence of Lemma 3.5 in [13]. \square

Lemma 2.3. *Let $H \in L^2(\Omega)$ and $H_{\mathbf{p}} = \langle H, \psi_{\mathbf{p}} \rangle$ be the Fourier coefficients of H for $\mathbf{p} = (p_1, \dots, p_d)$. Let's recall that $x_{i_k} = \frac{2i_k - 1}{2n_k}$, $k = \overline{1, d}$ then the following equality holds*

$$H_{\mathbf{p}} = \frac{\pi^d}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} H(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) - \overline{\Gamma}_{\mathbf{n}, \mathbf{p}} \quad (2.9)$$

where \mathbf{p} satisfies that $p_k = \overline{1, n_k - 1}$, $k = \overline{1, d}$. Here $\overline{\Gamma}_{\mathbf{n}, \mathbf{p}}$ is defined by

$$\overline{\Gamma}_{\mathbf{n}, \mathbf{p}} = \sum_{\mathbf{r} \in 2\mathbb{L} \cdot \mathbf{n} \pm \mathbf{p}} \mathbf{1}_{\mathbb{N}^d, l_1^2 + \dots, l_d^2 \neq 0} H_{\mathbf{r}}.$$

Proof. First, we have the expansion of the function $H \in L^2(\Omega)$ as the following Fourier series

$$H(\mathbf{x}) = \sum_{\mathbf{r} \in \mathbb{N}^d} H_{\mathbf{r}} \psi_{\mathbf{r}}(\mathbf{x}). \quad (2.10)$$

where $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{N}^d$. Plug $\mathbf{x} = (x_{i_1}, \dots, x_{i_d})$ into the latter equation, we obtain

$$H(x_{i_1}, \dots, x_{i_d}) = \sum_{\mathbf{r} \in \mathbb{N}^d} H_{\mathbf{r}} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}). \quad (2.11)$$

This implies that

$$\begin{aligned}
& \frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} H(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \\
&= \frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \left(\sum_{\mathbf{r} \in \mathbb{N}^d} H_{\mathbf{r}} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \right) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \\
&= \sum_{\mathbf{r} \in \mathbb{N}^d} H_{\mathbf{r}} \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right) \\
&= \underbrace{\sum_{\mathbf{r} \notin \mathcal{N}_d} H_{\mathbf{r}} \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right)}_{:= \mathbb{H}_1} \\
&+ \underbrace{\sum_{\mathbf{r} \in \mathcal{N}_d} H_{\mathbf{r}} \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right)}_{:= \mathbb{H}_2},
\end{aligned}$$

where we denote

$$\mathcal{N}_d := \left\{ \mathbf{r} = (r_1, r_2, \dots, r_d) \in \mathbb{N}^d : r_k = \overline{1, n_k - 1}, k = \overline{1, d} \right\}. \quad (2.12)$$

Step 1. Consider \mathbb{H}_1 .

If $\mathbf{r} \notin \mathcal{N}_d$ then applying the first part of Lemma 2.2, we have

$$\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) = \begin{cases} \frac{1}{\pi^d}, & r_k \mp p_k = 2l_k n_k, k = \overline{1, d}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.13)$$

Hence, we deduce that

$$\begin{aligned}
& \sum_{\mathbf{r} \notin \mathcal{N}_d} H_{\mathbf{r}} \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right) \\
&= \begin{cases} \frac{1}{\pi^d} \sum_{\mathbf{r} \notin \mathcal{N}_d} H_{\mathbf{r}}, & r_k \mp p_k = 2l_k n_k, k = \overline{1, d}, \\ 0, & \text{otherwise.} \end{cases} \\
&= \frac{1}{\pi^d} \sum_{\substack{\mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p} \\ \mathbf{r} \notin \mathcal{N}_d}} H_{\mathbf{r}}.
\end{aligned} \quad (2.14)$$

where noting that the equation $r_k \mp p_k = 2l_k n_k$ is equivalent to $\mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}$.

For the sum $\sum_{\mathbf{r} \notin \mathcal{N}_d}^{\mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}} H_{\mathbf{r}}$ on the right hand side of (2.14), since $\mathbf{r} \notin \mathcal{N}_d$, we can see that \mathbf{r} is not different than \mathbf{p} . This implies that $\mathbf{l} = (l_1, \dots, l_d) \in \mathbb{N}^d$ in the sum $\sum_{\mathbf{r} \notin \mathcal{N}_d}^{\mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}} H_{\mathbf{r}}$ satisfies the following condition

$$l_1^2 + \dots + l_d^2 \neq 0.$$

Therefore, we can rewrite (2.14) as follows

$$\sum_{\mathbf{r} \notin \mathcal{N}_d} H_{\mathbf{r}} \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right) = \frac{1}{\pi^d} \sum_{\substack{\mathbf{l} \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0 \\ \mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}}} H_{\mathbf{r}}. \quad (2.15)$$

Step 2. Consider \mathbb{H}_2 .

If $\mathbf{r} \in \mathcal{N}_d$ then applying the second part of Lemma 2.2, we get

$$\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_{\mathbf{r}}(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) = \begin{cases} \frac{1}{\pi^d}, & r_k = p_k, k = \overline{1, d}, \\ 0, & \exists k = \overline{1, d} \text{ } r_k \neq p_k, \end{cases} \quad (2.16)$$

This leads to

$$\sum_{r \in \mathcal{N}_d} H_r \left(\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \psi_r(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) \right) = \frac{1}{\pi^d} H_{\mathbf{p}}. \quad (2.17)$$

Combining (2.12), (2.15), (2.17), we obtain

$$\frac{1}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} H(x_{i_1}, \dots, x_{i_d}) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) = \frac{1}{\pi^d} \left(H_{\mathbf{p}} + \sum_{\mathbf{r}=2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}}^{l \in \mathbb{N}^d, l_1^2 + \dots, l_d^2 \neq 0} H_{\mathbf{r}} \right). \quad (2.18)$$

This completes the proof of this Lemma. \square

Using Lemma 2.3, we obtain the following Lemma

Lemma 2.4. *Assume that $G \in C([0, T]; C^1(\bar{\Omega}))$ then for $t \in [0, T]$, we have*

$$G_{\mathbf{p}}(t) = \frac{\pi^d}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} G(x_{i_1}, \dots, x_{i_d}, t) \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) - \bar{\Pi}_{\mathbf{n}, \mathbf{p}}(t) \quad (2.19)$$

where \mathbf{p} satisfies that $p_k = \overline{1, n_k - 1}$, $k = \overline{1, d}$ and

$$\bar{\Pi}_{\mathbf{n}, \mathbf{p}}(t) = \sum_{\mathbf{r}=2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}}^{l \in \mathbb{N}^d, l_1^2 + \dots, l_d^2 = 0} G_{\mathbf{r}}(t).$$

Next we consider the following Lemma

Lemma 2.5. *Assume that $\mu = (\mu_1, \dots, \mu_d)$. Let us choose $\mu_0 \geq d \max(\mu_1, \dots, \mu_d)$. Then if $H \in \mathcal{H}^{\mu_0}(\Omega)$ then*

$$|H_{\mathbf{p}}| = |H_{p_1, \dots, p_d}| \leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)} \prod_{k=1}^d p_k^{-\mu_k}, \quad \text{for } \mathbf{p} = (p_1, p_2, \dots, p_d). \quad (2.20)$$

Proof. Using the Cauchy inequality

$$(a_1 + \dots + a_d)^d \geq d \prod_{k=1}^d a_k$$

for any $a_k \geq 0, k = \overline{1, d}$, we have

$$\begin{aligned} \left(\sum_{k=1}^d p_k^2 \right)^{\frac{d \max(\mu_1, \dots, \mu_d)}{2}} &\geq \left(d \prod_{k=1}^d p_k^2 \right)^{\frac{\max(\mu_1, \dots, \mu_d)}{2}} \\ &\geq d^{\frac{\max(\mu_1, \dots, \mu_d)}{2}} \prod_{k=1}^d p_k^{\mu_k}. \end{aligned}$$

The left hand side of the latter inequality is bounded by

$$\left(\sum_{k=1}^d p_k^2 \right)^{\frac{d \max(\mu_1, \dots, \mu_d)}{2}} = |\mathbf{p}|^{d \max(\mu_1, \dots, \mu_d)} \leq |\mathbf{p}|^{\mu_0}$$

The observations above imply that

$$\prod_{k=1}^d p_k^{\mu_k} \leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} |\mathbf{p}|^{\mu_0}.$$

This leads to

$$\left| \prod_{k=1}^d p_k^{\mu_k} \langle H, \psi_{\mathbf{p}} \rangle \right| \leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} |\mathbf{p}|^{\mu_0} \langle H, \psi_{\mathbf{p}} \rangle \leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}$$

where we used the fact that

$$\left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2 = \sum_{p_1=1}^{\infty} \dots \sum_{p_d=1}^{\infty} |\mathbf{p}|^{2\mu_0} \langle H, \psi_{\mathbf{p}} \rangle^2 \geq |\mathbf{p}|^{2\mu_0} \langle H, \psi_{\mathbf{p}} \rangle^2. \quad (2.21)$$

This completes the proof of Lemma. \square

Proof of Theorem 2.1. Using Lemma 2.3 and by a simple computation, we get

$$\begin{aligned} & \left\| \widehat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 \\ &= 4 \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} \left[\frac{\pi^d}{\prod_{k=1}^d n_k} \sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \Lambda_{i_1, i_2, \dots, i_d} \Upsilon_{i_1, i_2, \dots, i_d} \psi_{\mathbf{p}}(x_{i_1}, \dots, x_{i_d}) - \bar{\Gamma}_{\mathbf{n}, \mathbf{p}} \right]^2 \\ & \quad + 4 \sum_{\mathbf{p} \notin \mathcal{W}_{\beta_{\mathbf{n}}}} |H_{\mathbf{p}}|^2 \\ &\leq \underbrace{\frac{8\pi^{2d}}{\left(\prod_{k=1}^d n_k\right)^2} \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} \left[\sum_{i_1=1}^{n_1} \dots \sum_{i_d=1}^{n_d} \Lambda_{i_1, i_2, \dots, i_d} \Upsilon_{i_1, i_2, \dots, i_d} \right]^2}_{:= A_{1,1}} \\ & \quad + 8 \underbrace{\sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} |\bar{\Gamma}_{\mathbf{n}, \mathbf{p}}|^2}_{:= A_{1,2}} + 4 \underbrace{\sum_{\mathbf{p} \notin \mathcal{W}_{\beta_{\mathbf{n}}}} |H_{\mathbf{p}}|^2}_{:= A_{1,3}} \end{aligned} \quad (2.22)$$

where we used the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. The expectation of $A_{1,1}$ is bounded by

$$\begin{aligned} \mathbf{E} A_{1,1} &\leq \frac{8\pi^{2d}}{\left(\prod_{k=1}^d n_k\right)^2} \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} \frac{\prod_{k=1}^d n_k}{\pi^d} V_{\max}^2 \\ &= \frac{8\pi^d}{\prod_{k=1}^d n_k} V_{\max}^2 \text{card}(\mathcal{W}_{\beta_{\mathbf{n}}}), \end{aligned} \quad (2.23)$$

above we used the fact that $\mathbf{E}^2 \Upsilon_{i_1, i_2, \dots, i_d} = 1$ and $\mathbf{E}(\Upsilon_{i_1, i_2, \dots, i_d} \Upsilon_{j_1, j_2, \dots, j_d}) = 0$. Now we estimate the

$$\text{card}(\mathcal{W}_{\beta_{\mathbf{n}}}) = \text{card} \left(\left\{ \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d : \sum_{k=1}^d p_k^2 \leq \beta_{\mathbf{n}} = \beta(n_1, \dots, n_d) \right\} \right), \quad (2.24)$$

which is the number of \mathbf{p} such that $|\mathbf{p}|^2 \leq \beta_{\mathbf{n}}$. Let any $\mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d$ such that $\sum_{k=1}^d p_k^2 \leq \beta_{\mathbf{n}}$. Let us define rectangles $\mathbf{Q}_{\mathbf{p}}$ in \mathbb{R}^d as follows

$$\mathbf{Q}_{\mathbf{p}} = \left\{ \mathbf{z} = (z_1, z_2, \dots, z_d) \in \mathbb{R}^d : p_k - 1 \leq z_k \leq p_k \right\}.$$

Let us define the set $\mathbf{Q}_{\sqrt{\beta_{\mathbf{n}}}}$ as follows

$$\mathbf{Q}_{\sqrt{\beta_{\mathbf{n}}}} = \bigcup_{\mathbf{p}^2 \leq \beta_{\mathbf{n}}} \mathbf{Q}_{\mathbf{p}}.$$

It is easy to see that $\text{card}(\mathcal{W}_{\beta_{\mathbf{n}}})$ is equal to the volume of the set $\mathbf{Q}_{\sqrt{\beta_{\mathbf{n}}}}$ and we can realize that

$$\mathbf{Q}_{\sqrt{\beta_{\mathbf{n}}}} \subset \mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}} = \left\{ \mathbf{z} = (z_1, \dots, z_d) \in \mathbb{R}^d : \sum_{k=1}^d z_k^2 \leq \beta_{\mathbf{n}} \right\}.$$

Hence $\text{card}(\mathcal{W}_{\beta_{\mathbf{n}}})$ is less than the volume of the set $\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}$ which denoted by $\text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}})$ i.e.,

$$\text{card}(\mathcal{W}_{\beta_{\mathbf{n}}}) \leq \text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}).$$

Now we find an upper bound of $\text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}})$. First, we have

$$\text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}) = \int_{\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}} \mathbf{d}z_1 \mathbf{d}z_2 \dots \mathbf{d}z_d. \quad (2.25)$$

We give the following coordinate system as follows

$$\begin{aligned} z_1 &= r \cos(\theta_1), \quad z_2 = r \sin(\theta_1) \cos(\theta_2), \quad z_3 = r \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \dots \\ z_{d-1} &= r \sin(\theta_1) \dots \sin(\theta_{d-2}) \cos(\theta_{d-1}) \\ z_d &= r \sin(\theta_1) \dots \sin(\theta_{d-2}) \sin(\theta_{d-1}) \end{aligned}$$

where

$$1 \leq r \leq \sqrt{\beta_{\mathbf{n}}}, \quad 0 \leq \theta_i \leq \pi, i = \overline{1, d-2}, \quad 0 \leq \theta_{d-1} < 2\pi.$$

From the Change of Variables formula that the rectangular volume element $\mathbf{d}z_1 \mathbf{d}z_2 \dots \mathbf{d}z_d$ can be written in spherical coordinates as

$$\begin{aligned} \mathbf{d}z_1 \mathbf{d}z_2 \dots \mathbf{d}z_d &= \left| \det \left(\frac{\partial x_i}{\partial (r, \theta_j)} \right) \right|_{1 \leq i \leq d, 1 \leq j \leq d-1} \mathbf{d}r \mathbf{d}\theta_1 \dots \mathbf{d}\theta_{d-1} \\ &= r^{d-1} \sin^{d-2}(\theta_1) \sin^{d-3}(\theta_2) \dots \sin(\theta_{d-2}) \mathbf{d}r \mathbf{d}\theta_1 \dots \mathbf{d}\theta_{d-1}. \end{aligned}$$

Hence, applying Fubini's theorem, we obtain that

$$\begin{aligned} &\text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}) \\ &= \int_{\mathcal{W}_{\rho_{\mathbf{n}}}(\mathbf{n})} \mathbf{d}z_1 \mathbf{d}z_2 \dots \mathbf{d}z_d \\ &= \int_0^{2\pi} \int_0^\pi \dots \int_0^\pi \int_1^{\sqrt{\beta_{\mathbf{n}}}} r^{d-1} \sin^{d-2}(\theta_1) \sin^{d-3}(\theta_2) \dots \sin(\theta_{d-2}) \mathbf{d}r \mathbf{d}\theta_1 \dots \mathbf{d}\theta_{d-2} \mathbf{d}\theta_{d-1} \\ &= \left(\int_0^{2\pi} \mathbf{d}\theta_{d-1} \right) \left(\int_1^{\sqrt{\beta_{\mathbf{n}}}} r^{d-1} \mathbf{d}r \right) \left(\int_0^\pi \sin^{d-2}(\theta_1) \mathbf{d}\theta_1 \right) \left(\int_0^\pi \sin^{d-3}(\theta_2) \mathbf{d}\theta_2 \right) \dots \left(\int_0^\pi \sin(\theta_{d-2}) \mathbf{d}\theta_{d-2} \right) \\ &= \frac{2\pi \left(\beta_{\mathbf{n}}^{d/2} - 1 \right)}{d} \left(\int_0^\pi \sin^{d-2}(\theta_1) \mathbf{d}\theta_1 \right) \left(\int_0^\pi \sin^{d-3}(\theta_2) \mathbf{d}\theta_2 \right) \dots \left(\int_0^\pi \sin(\theta_{d-2}) \mathbf{d}\theta_{d-2} \right). \end{aligned} \quad (2.26)$$

Thanks to page 245, [15], we know that

$$\left(\int_0^\pi \sin^{d-2}(\theta_1) \mathbf{d}\theta_1 \right) \left(\int_0^\pi \sin^{d-3}(\theta_2) \mathbf{d}\theta_2 \right) \dots \left(\int_0^\pi \sin(\theta_{d-2}) \mathbf{d}\theta_{d-2} \right) = \frac{d}{2\pi} \text{Vol}(\mathbb{B}^d(1)). \quad (2.27)$$

Here $\text{Vol}(\mathbb{B}^d(1))$ denotes the volume of the unit d -ball. Using again [15] (Proposition 4.2, page 246-247), we obtain that

$$\text{Vol}(\mathbb{B}^d(1)) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}. \quad (2.28)$$

Combining (2.26), (2.27), (2.28) gives

$$\text{Vol}(\mathbf{Q}'_{\sqrt{\beta_{\mathbf{n}}}}) \leq \frac{\pi^{d/2} \beta_{\mathbf{n}}^{d/2}}{\Gamma(\frac{d}{2} + 1)} = \frac{2\pi^{d/2}}{d\Gamma(d/2)} \beta_{\mathbf{n}}^{d/2} \quad (2.29)$$

which we have used the fact that $\Gamma(\frac{d}{2} + 1) = \frac{d}{2} \Gamma(\frac{d}{2})$. This together with (2) leads to

$$\begin{aligned} \text{card}(\mathcal{W}_{\beta_{\mathbf{n}}}) &= \text{card} \left(\left\{ \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d, \quad p_k \geq 1, k = \overline{1, d} : \sum_{k=1}^d p_k^2 \leq \beta_{\mathbf{n}} = \beta(n_1, \dots, n_d) \right\} \right) \\ &\leq \frac{2\pi^{d/2}}{d\Gamma(d/2)} \beta_{\mathbf{n}}^{d/2}. \end{aligned} \quad (2.30)$$

This implies that

$$\mathbf{E}A_{1,1} \leq 8\pi^d V_{max}^2 \frac{2\pi^{d/2}}{d\Gamma(d/2)} \frac{\beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d n_k}. \quad (2.31)$$

Next, in order to estimate $A_{1,2}$, we need to find an upper bound of $\overline{\Gamma}_{\mathbf{n},\mathbf{p}}$.

Using Lemma 2.5, we estimate $|\overline{\Gamma}_{\mathbf{n},\mathbf{p}}|$ as follows

$$\begin{aligned} |\overline{\Gamma}_{\mathbf{n},\mathbf{p}}| &\leq \sum_{\substack{l \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0 \\ \mathbf{r} = 2\mathbf{l} \cdot \mathbf{n} \pm \mathbf{p}}} |H_{\mathbf{r}}| = \sum_{l \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0} \left| H_{2l_1 n_1 \pm p_1, \dots, 2l_d n_d \pm p_d} \right| \\ &\leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)} \sum_{l \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0} \prod_{k=1}^d (2l_k n_k \pm p_k)^{-\mu_k}. \end{aligned} \quad (2.32)$$

Furthermore, for $p_k = \overline{1, n_k}$ for $k = \overline{1, d}$, we have

$$\prod_{k=1}^d (2l_k n_k + p_k)^{\mu_k} \geq \prod_{k=1}^d (n_k)^{\mu_k} \prod_{k=1}^d (2l_k)^{\mu_k} \quad (2.33)$$

and

$$\prod_{k=1}^d (2l_k n_k - p_k)^{\mu_k} \geq \prod_{k=1}^d (2l_k n_k - n_k)^{\mu_k} \geq \prod_{k=1}^d (n_k)^{\mu_k} \prod_{k=1}^d (2l_k - 1)^{\mu_k}. \quad (2.34)$$

Combining (2.33) and (2.34) gives

$$\prod_{k=1}^d (2l_k n_k \pm p_k)^{-\mu_k} \leq \prod_{k=1}^d (n_k)^{-2\mu_k} \prod_{k=1}^d (2l_k - 1)^{-2\mu_k}. \quad (2.35)$$

This together with (2.32) implies that

$$|\overline{\Gamma}_{\mathbf{n},\mathbf{p}}| \leq d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)} \prod_{k=1}^d (n_k)^{-2\mu_k} \sum_{l \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0} \prod_{k=1}^d (2l_k - 1)^{-2\mu_k}. \quad (2.36)$$

Since $\mu_k > \frac{1}{2}$, we know that the series $d^{-\frac{\max(\mu_1, \dots, \mu_d)}{2}} \sum_{l \in \mathbb{N}^d, l_1^2 + \dots + l_d^2 \neq 0} \prod_{k=1}^d (2l_k - 1)^{-2\mu_k}$ converges. So we define this sum to be $\mathcal{C}(\mu_1, \dots, \mu_d)$. Hence

$$|\overline{\Gamma}_{\mathbf{n},\mathbf{p}}| \leq \mathcal{C}(\mu_1, \dots, \mu_d) \|H\|_{\mathcal{H}^{\mu_0}(\Omega)} \prod_{k=1}^d (n_k)^{-2\mu_k}. \quad (2.37)$$

It follows from (2.30) that

$$\begin{aligned} A_{1,2} &= 8 \sum_{\mathbf{p} \in \mathcal{W}_{\beta_{\mathbf{n}}}} |\overline{\Gamma}_{\mathbf{n},\mathbf{p}}|^2 \\ &\leq 8\mathcal{C}^2(\mu_1, \dots, \mu_d) \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2 \prod_{k=1}^d (n_k)^{-4\mu_k} \text{card}(\mathcal{W}_{\beta_{\mathbf{n}}}) \\ &\leq \frac{16\mathcal{C}^2(\mu_1, \dots, \mu_d) \pi^{d/2}}{d\Gamma(d/2)} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2 \beta_{\mathbf{n}}^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k}. \end{aligned} \quad (2.38)$$

For $A_{1,3}$ on the right hand side of (2.22), noting that $|\mathbf{p}|^2 \geq \beta_{\mathbf{n}}$ if $\mathbf{p} \notin \mathcal{W}_{\beta_{\mathbf{n}}}$, we have the following estimate

$$A_{1,3} = 4 \sum_{\mathbf{p} \notin \mathcal{W}_{\beta_{\mathbf{n}}}} |\mathbf{p}|^{-2\mu_0} |\mathbf{p}|^{2\mu_0} |H_{\mathbf{p}}|^2 \leq 4\beta_{\mathbf{n}}^{-\mu_0} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2. \quad (2.39)$$

Combining (2.22), (2.23), (2.38), (2.39), we obtain

$$\begin{aligned}
& \mathbf{E} \left\| \widehat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 \\
& \leq \mathbf{E} A_{1,1} + A_{1,2} + A_{1,3} \\
& \leq 8\pi^d V_{max}^2 \frac{2\pi^{d/2}}{d2^d \Gamma(d/2)} \frac{\beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d n_k} + \frac{16C^2(\mu_1, \dots, \mu_d) \pi^{d/2}}{d\Gamma(d/2)} \|H\|_{\mathcal{H}^{\mu_0}(2\Omega)}^2 \beta_{\mathbf{n}}^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k} \\
& \quad + 4\beta_{\mathbf{n}}^{-\mu_0} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2 \\
& \leq \overline{C}(\mu_1, \dots, \mu_d, H) \beta_{\mathbf{n}}^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k} + 4\beta_{\mathbf{n}}^{-\mu_0} \|H\|_{\mathcal{H}^{\mu_0}(\Omega)}^2.
\end{aligned}$$

By a similar method used in the first part of the proof we can apply the previous results using Lemma 2.4, we immediately obtain the second error estimation. \square

3. BACKWARD PROBLEM FOR PARABOLIC EQUATION WITH CONSTANT COEFFICIENTS

In this section, we consider the problem of recovering $\mathbf{u}(\mathbf{x}, t)$, $0 \leq t < T$, such that

$$\begin{cases} \mathbf{u}_t(\mathbf{x}, t) + \mathcal{A}\mathbf{u}(\mathbf{x}, t) = F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t), & 0 < t < T, \quad \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), \end{cases} \quad (3.1)$$

where $H \in L^2(\Omega)$ is a given function. The operator \mathcal{A} solves the following eigenvalue problem

$$\mathcal{A}\psi_{\mathbf{p}}(\mathbf{x}) = \mathbf{M}(|\mathbf{p}|)\psi_{\mathbf{p}}(\mathbf{x}),$$

for a non-decreasing function \mathbf{M} and the eigenfunctions $\psi_{\mathbf{p}}(\mathbf{x})$ defined in (2.1).

Now, we give some examples of operators \mathcal{A} defined by the spectral theorem using the Laplacian in Ω and the corresponding eigenvalues.

Example 3.1. (a) If $\mathcal{A} = -\Delta$ in Ω with Dirichlet boundary conditions then Problem (3.1) is called nonlinear heat equation and its eigenvalues are

$$\mathbf{M}(|\mathbf{p}|) = |\mathbf{p}|^2 = p_1^2 + \dots + p_d^2.$$

In this case the eigenfunctions are given by equation (2.1).

(b) If $\mathcal{A} = \Delta^2$ in Ω with Dirichlet boundary conditions then Problem (3.1) is called nonlinear biharmonic heat equation (see [27]) and using the spectral theorem its eigenvalues are

$$\mathbf{M}(|\mathbf{p}|) = |\mathbf{p}|^4 = (p_1^2 + \dots + p_d^2)^2.$$

In this case again, the eigenfunctions are given by equation (2.1).

(c) If $\mathcal{A} = -\Delta + \Delta^2$ in Ω with Dirichlet boundary conditions then Problem (3.1) is called extended Fisher-Kolmogorov equation (see [19, 21]) and using the spectral theorem its eigenvalues are

$$\mathbf{M}(|\mathbf{p}|) = |\mathbf{p}|^2 + |\mathbf{p}|^4 = p_1^2 + \dots + p_d^2 + (p_1^2 + \dots + p_d^2)^2.$$

In this case again, the eigenfunctions are given by equation (2.1).

(d) If $\mathcal{A} = 2\Delta + \Delta^2$ in Ω with Dirichlet boundary conditions then Problem (4.1) is called Swift-Hohenberg equation (see [14]) and using the spectral theorem its eigenvalues are

$$\mathbf{M}(|\mathbf{p}|) = -2|\mathbf{p}|^2 + |\mathbf{p}|^4 = -2(p_1^2 + \dots + p_d^2) + (p_1^2 + \dots + p_d^2)^2.$$

In this case again, the eigenfunctions are given by equation (2.1).

Proposition 3.1. If Problem (3.1) has a unique solution \mathbf{u} then it satisfies that

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{p} \in \mathbb{N}^d} \left[e^{(T-t)\mathbf{M}(|\mathbf{p}|)} H_{\mathbf{p}} - \int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} G_{\mathbf{p}}(\tau) d\tau \right] \psi_{\mathbf{p}}(\mathbf{x})$$

$$- \sum_{\mathbf{p} \in \mathbb{N}^d} \left[\int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} F_{\mathbf{p}}(u)(\tau) d\tau \right] \psi_{\mathbf{p}}(\mathbf{x}). \quad (3.2)$$

Where $G_{\mathbf{p}}(\tau) = \langle G(\cdot, \tau), \psi_{\mathbf{p}} \rangle$, $H_{\mathbf{p}} = \langle H, \psi_{\mathbf{p}} \rangle$, $F_{\mathbf{p}}(\mathbf{u})(\tau) = \langle F(\mathbf{u}(\cdot, \tau)), \psi_{\mathbf{p}} \rangle$ are the Fourier coefficients.

Proof. The proof is similar to Theorem 3.1, page 2975, [39] and we omit it here. \square

Let's define $\mathbb{P}_{\overline{M}}$ as the operator of orthogonal projection onto the eigenspace spanned by the set $\{\psi_{\mathbf{p}}(\mathbf{x}), 1 \leq |\mathbf{p}|^2 \leq \overline{M}\}$ for a given \overline{M} i.e.,

$$\mathbb{P}_{\overline{M}} v(\mathbf{x}) := \sum_{|\mathbf{p}|^2 \leq \overline{M}} \left\langle v, \psi_{\mathbf{p}} \right\rangle_{L^2(\Omega)} \psi_{\mathbf{p}}(\mathbf{x}), \text{ for all } v \in L^2(\Omega). \quad (3.3)$$

Our method in this subsection is described as follows: First, we approximate the two functions H and G by $H_{\beta_{\mathbf{n}}}$ and $G_{\beta_{\mathbf{n}}}$ which are defined in Theorem (2.1). Then we use the Fourier truncation method by adding the operator $\mathbb{P}_{\rho_{\mathbf{n}}}$ and introducing the following regularized problem

$$\begin{cases} \frac{\partial \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t)}{\partial t} + \mathcal{A} \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) = \mathbb{P}_{\rho_{\mathbf{n}}} F(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t)) + \mathbb{P}_{\rho_{\mathbf{n}}} G(\mathbf{x}, t), & 0 < t < T, \\ \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, T) = \mathbb{P}_{\rho_{\mathbf{n}}} \hat{H}_{\beta_{\mathbf{n}}}(\mathbf{x}), \end{cases} \quad (3.4)$$

where $\rho_{\mathbf{n}}$ is called the regularization parameter. The function $\rho : \mathbb{N}^d \rightarrow \mathbb{R}$ is a function that depend on $\beta_{\mathbf{n}}$ and we use the notation $\rho_{\mathbf{n}} = \rho(\mathbf{n})$. Noting that $\lim_{\mathbf{n} \rightarrow 0} \rho_{\mathbf{n}} = \infty$. Define the set $\mathcal{W}_{\rho_{\mathbf{n}}}$ for any $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$

$$\mathcal{W}_{\rho_{\mathbf{n}}} = \left\{ \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{N}^d : |\mathbf{p}|^2 = \sum_{k=1}^d p_k^2 \leq \rho_{\mathbf{n}} = \rho(n_1, \dots, n_d) \right\}. \quad (3.5)$$

Since the solution of Problem (3.4) depends on two terms $\beta_{\mathbf{n}}$ and $\rho_{\mathbf{n}}$, we denote it by $\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}$.

Theorem 3.2. Suppose that $\beta_{\mathbf{n}} := \beta(n_1, n_2, \dots, n_d)$, $\rho_{\mathbf{n}} := \rho(n_1, n_2, \dots, n_d)$ are such that

$$\lim_{|\mathbf{n}| \rightarrow +\infty} \beta_{\mathbf{n}} = \lim_{|\mathbf{n}| \rightarrow +\infty} \rho_{\mathbf{n}} = +\infty, \quad \lim_{|\mathbf{n}| \rightarrow +\infty} \frac{e^{2TM(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}} = \lim_{|\mathbf{n}| \rightarrow +\infty} e^{2TM(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0} = 0, \quad (3.6)$$

where we recall $|\mathbf{n}| = \sqrt{\sum_{k=1}^d n_k^2}$. Assume that $H, G, H_{\beta_{\mathbf{n}}}, G_{\beta_{\mathbf{n}}}$ are as in Theorem 2.1. Suppose that $F \in L^\infty(\mathbb{R})$ and F is a Lipschitz function, i.e. there exists a positive constant K such that

$$|F(\xi_1) - F(\xi_2)| \leq K |\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \quad (3.7)$$

The Problem (3.4) has a unique solution $\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} \in C([0, T]; L^2(\Omega))$ which satisfies

$$\begin{aligned} \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) &= \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[e^{(T-t)\mathbf{M}(|\mathbf{p}|)} \hat{H}_{\beta_{\mathbf{n}}, \mathbf{p}} - \int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} \hat{G}_{\beta_{\mathbf{n}}, \mathbf{p}}(\tau) d\tau \right] \psi_{\mathbf{p}}(\mathbf{x}) \\ &\quad - \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[\int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} F_{\mathbf{p}}(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}})(\tau) d\tau \right] \psi_{\mathbf{p}}(\mathbf{x}). \end{aligned} \quad (3.8)$$

where $\hat{G}_{\beta_{\mathbf{n}}, \mathbf{p}}(t) = \langle \hat{G}_{\beta_{\mathbf{n}}}(\cdot, t), \psi_{\mathbf{p}} \rangle$ and $\hat{H}_{\beta_{\mathbf{n}}, \mathbf{p}} = \langle \hat{H}_{\beta_{\mathbf{n}}}, \psi_{\mathbf{p}} \rangle$.

(a) Assume that the problem (3.1) has a unique solution \mathbf{u} such that

$$\sum_{\mathbf{p} \in \mathbb{N}^d} e^{2t\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2(t) := \tilde{A}_1 < \infty, \text{ for all } t \in [0, T]. \quad (3.9)$$

Then as $|\mathbf{n}| \rightarrow \infty$, $\mathbf{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \max \left(\frac{e^{2TM(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2TM(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0}, 1 \right). \quad (3.10)$$

(b) Assume that the problem (3.1) has unique solution \mathbf{u} such that

$$\sum_{\mathbf{p} \in \mathbb{N}^d} |\mathbf{M}(|\mathbf{p}|)|^\alpha e^{2t\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2(t) := \tilde{A}_2 < \infty, \quad (3.11)$$

for any $\alpha > 0$ and $t \in [0, T]$. Then as $|\mathbf{n}| \rightarrow \infty$, $\mathbf{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \max \left(\frac{e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0}, |\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})|^{-2\alpha} \right). \quad (3.12)$$

for all $t \in [0, T]$.

(c) Assume that the problem (3.1) has a unique solution \mathbf{u} such that

$$\sum_{\mathbf{p} \in \mathbb{N}^d} e^{2(t+\delta)\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2 = \tilde{A}_3 < \infty, \quad (3.13)$$

for any real number $\delta \geq 0$ and $t \in [0, T]$. Then as $|\mathbf{n}| \rightarrow \infty$, $\mathbf{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \max \left(\frac{e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0}, e^{-2\delta\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \right). \quad (3.14)$$

Proof of Theorem 3.2. We divide the proof into some smaller parts.

Part 1. The nonlinear integral equation (3.8) has unique solution $\overline{U}_{\rho_{\mathbf{n}}} \in C([0, T]; L^2(\Omega))$.

The proof is similar to [38] (See Theorem 3.1, page 2975 [39]). Hence, we omit it here.

Part 2. The error estimate $\mathbf{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2}$.

First, using Parseval's identity, equations (3.2), and (3.8) we get

$$\begin{aligned} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2(\Omega)}^2 &\leq 4 \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[e^{(T-t)\mathbf{M}(|\mathbf{p}|)} \left(\widehat{H}_{\beta_{\mathbf{n}}, \mathbf{p}} - H_{\mathbf{p}} \right) \right]^2 \\ &\quad + 4 \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[\int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} \left(\widehat{G}_{\beta_{\mathbf{n}}, \mathbf{p}}(\tau) - G_{\mathbf{p}}(\tau) \right) d\tau \right]^2 \\ &\quad + 4 \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[\int_t^T e^{(\tau-t)\mathbf{M}(|\mathbf{p}|)} \left(F_{\mathbf{p}}(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}) - F_{\mathbf{p}}(\mathbf{u})(s) \right) ds \right]^2 \\ &\quad + 4 \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} |\mathbf{u}_{\mathbf{p}}|^2. \end{aligned} \quad (3.15)$$

Using the Cauchy-Schwartz inequality, the expectation of the right hand side of (3.15) is bounded by

$$\begin{aligned} &\mathbf{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2(\Omega)}^2 \\ &\leq 4e^{2(T-t)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \mathbf{E} \left(\sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[\left(\widehat{H}_{\beta_{\mathbf{n}}, \mathbf{p}} - H_{\mathbf{p}} \right) \right]^2 \right) \\ &\quad + 4 \int_t^T e^{2(\tau-t)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} d\tau \mathbf{E} \left(\sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \left[\int_t^T \left(\widehat{G}_{\beta_{\mathbf{n}}, \mathbf{p}}(\tau) - G_{\mathbf{p}}(\tau) \right)^2 d\tau \right] \right) \\ &\quad + 4(T-t) \int_t^T e^{2(\tau-t)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \sum_{\mathbf{p} \in \mathcal{W}_{\rho_{\mathbf{n}}}} \mathbf{E} \left(\left[\left(F_{\mathbf{p}}(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}})(\tau) - F_{\mathbf{p}}(\mathbf{u})(\tau) \right) \right]^2 \right) ds \\ &\quad + 4 \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} |\mathbf{u}_{\mathbf{p}}|^2. \end{aligned} \quad (3.16)$$

It follows from the Lipschitz property of F that

$$\begin{aligned}
\mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n} - \mathbf{u} \right\|_{L^2(\Omega)}^2 &\leq 4e^{2(T-t)\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 \\
&\quad + 4Te^{2(T-t)\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \widehat{G}_{\beta_n} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
&\quad + 4TK^2 \int_t^T e^{2(\tau-t)\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, \tau) - \mathbf{u}(\cdot, \tau) \right\|_{L^2(\Omega)}^2 d\tau \\
&\quad + 4 \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_n}} |\mathbf{u}_{\mathbf{p}}|^2.
\end{aligned} \tag{3.17}$$

Now, we deal with the three cases.

Case 1. Assume that the series

$$\sum_{\mathbf{p} \in \mathbb{N}^d} e^{2t\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2 \tag{3.18}$$

converges to \widetilde{A}_1 . Then multiplying both sides of the inequality (3.17) by $e^{2t\mathbf{M}(\sqrt{\rho_n})}$, we obtain that

$$\begin{aligned}
&e^{2t\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 \\
&\leq 4e^{2T\mathbf{M}(\sqrt{\rho_n})} \left(\mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T\mathbf{E} \left\| \widehat{G}_{\beta_n} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) + 4\widetilde{A}_1 \\
&\quad + 4TK^2 \int_t^T e^{2\tau\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, \tau) - \mathbf{u}(\cdot, \tau) \right\|_{L^2(\Omega)}^2 d\tau,
\end{aligned}$$

where we used the fact that

$$\sum_{\mathbf{p} \notin \mathcal{W}_{\rho_n}} |\mathbf{u}_{\mathbf{p}}|^2 = \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_n}} e^{-2t\mathbf{M}(|\mathbf{p}|)} e^{2t\mathbf{M}(|\mathbf{p}|)} |\mathbf{u}_{\mathbf{p}}|^2 \leq e^{-2t\mathbf{M}(\sqrt{\rho_n})} \widetilde{A}_1.$$

Above we used the monotone increasing property of \mathbf{M} .

Using the above Gronwall's inequality, we obtain that

$$\begin{aligned}
&e^{2t\mathbf{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n} - \mathbf{u} \right\|_{L^2(\Omega)}^2 \\
&\leq 4e^{2T\mathbf{M}(\sqrt{\rho_n})} \left(\mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T\mathbf{E} \left\| \widehat{G}_{\beta_n} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) e^{4TK^2(T-t)} \\
&\quad + 4\widetilde{A}_1 e^{4TK^2(T-t)}.
\end{aligned} \tag{3.19}$$

This implies that

$$\begin{aligned}
&\mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n} - \mathbf{u} \right\|_{L^2(\Omega)}^2 \\
&\leq \underbrace{4e^{4TK^2(T-t)}}_{C5^2} e^{2(T-t)\mathbf{M}(\sqrt{\rho_n})} \left(\mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T\mathbf{E} \left\| \widehat{G}_{\beta_n} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \\
&\quad + 4\widetilde{A}_1 e^{4TK^2(T-t)} e^{-2t\mathbf{M}(\sqrt{\rho_n})}.
\end{aligned} \tag{3.20}$$

It follows from Corollary (2.1) that $\mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_n})} \max \left(\frac{e^{2T\mathbf{M}(\sqrt{\rho_n})} \beta_n^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2T\mathbf{M}(\sqrt{\rho_n})} \beta_n^{-\mu_0}, 1 \right). \tag{3.21}$$

Case 2. Suppose that the series

$$\sum_{\mathbf{p} \in \mathbb{N}^d} \left| \mathbf{M}(|\mathbf{p}|) \right|^{2\alpha} e^{2t\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2 \tag{3.22}$$

converges to \tilde{A}_2 . By a similar technique as in case 1 above and using the following estimate

$$\begin{aligned} \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} |\mathbf{u}_{\mathbf{p}}|^2 &= \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} \left| \mathbf{M}(|\mathbf{p}|) \right|^{-2\alpha} e^{-2t\mathbf{M}(|\mathbf{p}|)} \left| \mathbf{M}(|\mathbf{p}|) \right|^{2\alpha} e^{2t\mathbf{M}(|\mathbf{p}|)} |\mathbf{u}_{\mathbf{p}}|^2 \\ &\leq \left| \mathbf{M}(\sqrt{\rho_{\mathbf{n}}}) \right|^{-2\alpha} e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \tilde{A}_2, \end{aligned} \quad (3.23)$$

we deduce that

$$\begin{aligned} &\mathbf{E} \left\| \bar{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2(\Omega)}^2 \\ &\leq \underbrace{4e^{4TK^2(T-t)}}_{C5^2} e^{2(T-t)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \left(\mathbf{E} \left\| \hat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \hat{G}_{\beta_{\mathbf{n}}} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \\ &+ 4\tilde{A}_2 e^{4TK^2(T-t)} \left| \mathbf{M}(\sqrt{\rho_{\mathbf{n}}}) \right|^{-2\alpha} e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})}. \end{aligned} \quad (3.24)$$

It follows from Corollary (2.1) that $\mathbf{E} \left\| \bar{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \max \left(\frac{e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0}, \left| \mathbf{M}(\sqrt{\rho_{\mathbf{n}}}) \right|^{-2\alpha} \right). \quad (3.25)$$

Case 3. Suppose that the series

$$\sum_{\mathbf{p} \in \mathbb{N}^d} e^{2(t+\delta)\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2$$

converges to \tilde{A}_3 . By a similar technique as in case 1 above and using the following estimate

$$\begin{aligned} \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} |\mathbf{u}_{\mathbf{p}}|^2 &= \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_{\mathbf{n}}}} e^{-2(t+\delta)\mathbf{M}(|\mathbf{p}|)} e^{2(t+\delta)\mathbf{M}(|\mathbf{p}|)} \mathbf{u}_{\mathbf{p}}^2 \\ &\leq e^{-2(t+\delta)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \tilde{A}_3, \end{aligned} \quad (3.26)$$

we deduce that

$$\begin{aligned} &\mathbf{E} \left\| \bar{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2(\Omega)}^2 \\ &\leq \underbrace{4e^{4TK^2(T-t)}}_{C5^2} e^{2(T-t)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \left(\mathbf{E} \left\| \hat{H}_{\beta_{\mathbf{n}}} - H \right\|_{L^2(\Omega)}^2 + \mathbf{E} \left\| \hat{G}_{\beta_{\mathbf{n}}} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \\ &+ 4\tilde{A}_3 e^{4TK^2(T-t)} e^{-2(t+\delta)\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})}. \end{aligned}$$

It follows from Corollary (2.1) that $\mathbf{E} \left\| \bar{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} - \mathbf{u} \right\|_{L^2(\Omega)}^2$ is of order

$$e^{-2t\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \max \left(\frac{e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{-\mu_0}, e^{-2\delta\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \right). \quad (3.27)$$

□

Remark 3.1. We give one choice for $\beta_{\mathbf{n}}$ and $\rho_{\mathbf{n}}$ which satisfies (3.6). Let $0 < 2\alpha_0 < \mu_0$ and $e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} = \beta_{\mathbf{n}}^{2\alpha_0}$. Since $\frac{e^{2T\mathbf{M}(\sqrt{\rho_{\mathbf{n}}})} \beta_{\mathbf{n}}^{d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}} \rightarrow 0$ when $|\mathbf{n}| \rightarrow +\infty$, we can choose $\beta_{\mathbf{n}}$ such that

$$\lim_{|\mathbf{n}| \rightarrow +\infty} \frac{\beta_{\mathbf{n}}^{2\alpha_0 + d/2}}{\prod_{k=1}^d (n_k)^{4\mu_k}} = 0. \quad (3.28)$$

Let us choose $\beta_{\mathbf{n}}^{2\alpha_0 + d/2} = \prod_{k=1}^d n_k$ then $\beta_{\mathbf{n}} = \left(\prod_{k=1}^d n_k \right)^{\frac{1}{2\alpha_0 + d/2}}$ and then we choose $\rho_{\mathbf{n}}$ such that

$$\mathbf{M}(\sqrt{\rho_{\mathbf{n}}}) = \frac{\alpha_0}{T} \log(\beta_{\mathbf{n}}) = \frac{\alpha_0}{T(2\alpha_0 + d/2)} \log \left(\prod_{k=1}^d n_k \right). \quad (3.29)$$

In above theorem, for the case (b), $\mathbf{E} \left\| \bar{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$\left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 t}{4T\alpha_0 + dT}} \max \left(\frac{1}{\prod_{k=1}^d (n_k)^{4\mu_k - 1}}, \left(\prod_{k=1}^d n_k \right)^{\frac{2\alpha_0 - \mu_0}{2\alpha_0 + d/2}}, \log^{-2\alpha} \left(\prod_{k=1}^d n_k \right) \right). \quad (3.30)$$

In above theorem, for the case (c), $\mathbf{E} \left\| \bar{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2$ is of order

$$\left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 t}{4T\alpha_0 + dT}} \max \left(\frac{1}{\prod_{k=1}^d (n_k)^{4\mu_k - 1}}, \left(\prod_{k=1}^d n_k \right)^{\frac{2\alpha_0 - \mu_0}{2\alpha_0 + d/2}}, \left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 \delta}{4T\alpha_0 + dT}} \right). \quad (3.31)$$

4. THE BACKWARD PROBLEM FOR PARABOLIC EQUATION WITH TIME DEPENDENT COEFFICIENTS

4.1. The problem with coefficients that depend only on t . In this section, we consider the problem of constructing a solution $\mathbf{u} \in C([0, T; \mathcal{V}(\Omega)])$, $\mathbf{u}' \in L^2(0, T; L^2(\Omega))$ such that \mathbf{u} satisfies the following parabolic equation with time dependent coefficients

$$\begin{cases} \mathbf{u}_t + \mathcal{A}(t)\mathbf{u} = F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t), & \mathbf{x} \in \Omega, 0 < t < T, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases} \quad (4.1)$$

where $H \in L^2(\Omega)$. Here $\mathcal{V}(\Omega) \xhookrightarrow{d} L^2(\Omega)$; i.e., $\mathcal{V}(\Omega) \subset L^2(\Omega)$ is continuously embedded into $L^2(\Omega)$. It means that there exists some constant $m_0 > 0$ such that for all $v \in \mathcal{V}(\Omega)$

$$\|v\|_{L^2(\Omega)} \leq m_0 \|v\|_{\mathcal{V}(\Omega)}. \quad (4.2)$$

Then $L^2(\Omega) \xhookrightarrow{d} \mathcal{V}'(\Omega)$ via $v \mapsto \langle v | \cdot \rangle_{L^2(\Omega)}$, where $\mathcal{V}'(\Omega)$ denotes the dual space of $\mathcal{V}(\Omega)$. In this section, we assume that the source function $F: \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function i.e, for each $Q > 0$ and for any u, v satisfying $|u|, |v| \leq Q$, there holds

$$|F(u) - F(v)| \leq K_F(Q) |u - v|, \quad (4.3)$$

where

$$K(Q) := \sup \left\{ \left| \frac{F(u) - F(v)}{u - v} \right| : |u|, |v| \leq Q, u \neq v \right\} < +\infty. \quad (4.4)$$

We note that the function $Q \rightarrow K(Q)$ is increasing and $\lim_{Q \rightarrow +\infty} K(Q) = +\infty$.

First, we give the following definition

Definition 4.1. The pair of operators $(\mathcal{A}(t), \mathbf{P})$ satisfies Assumption (A) if the following conditions hold

(a) For any $v \in L^2(\Omega)$, there exists an increasing function $\bar{M}: \mathbb{R} \rightarrow \mathbb{R}^+$ such as

$$\mathbf{P}v(\mathbf{x}, t) = \sum_{\mathbf{p} \in \mathbb{N}^d} \bar{M}(|\mathbf{p}|) \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)} \psi_{\mathbf{p}}(\mathbf{x}), \quad v \in L^2(\Omega). \quad (4.5)$$

(b) For $u \in \mathcal{V}(\Omega), v \in \mathcal{V}(\Omega)$

$$\left\langle (\mathbf{P} - \mathcal{A}(t))u, v \right\rangle_{L^2(\Omega)} \leq \widetilde{M}_a \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)}, \quad (4.6)$$

for some constant $\widetilde{M}_a > 0$.

(c) For any $v \in \mathcal{V}(\Omega)$, there exists $\widehat{M} > 0$ such that

$$\left\langle (\mathbf{P} - \mathcal{A}(t))v, v \right\rangle_{L^2(\Omega)} \geq \widehat{M} \|v\|_{\mathcal{V}(\Omega)}^2. \quad (4.7)$$

Now we state the following lemma concerning an estimate of \mathbf{P}_{ρ_n} .

Lemma 4.1. Let \mathbf{P}_{ρ_n} be defined as follows

$$\mathbf{P}_{\rho_n} v(\mathbf{x}) = \sum_{\mathbf{p} \in \mathcal{W}_{\rho_n}} \overline{M}(|\mathbf{p}|) \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)} \psi_{\mathbf{p}}(\mathbf{x}), \quad \text{for all } v \in L^2(\Omega). \quad (4.8)$$

The operator \mathbf{P}_{ρ_n} is a linear, bounded operator, and satisfies that

$$\|\mathbf{P}_{\rho_n}\|_{\mathbb{L}(L^2(\Omega); L^2(\Omega))} \leq \overline{M}(\sqrt{\rho_n}), \quad (4.9)$$

where $\mathbb{L}(L^2(\Omega); L^2(\Omega))$ is the space of all bounded linear operators from $L^2(\Omega)$ to $L^2(\Omega)$.

Proof. For $v \in L^2(\Omega)$, since \overline{M} is a non-decreasing function, we have

$$\begin{aligned} \|\mathbf{P}_{\rho_n} v\|_{L^2(\Omega)}^2 &= \sum_{\mathbf{p} \in \mathcal{W}_{\rho_n}} \left| \overline{M}(|\mathbf{p}|) \right|^2 \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2 \\ &\leq \left| \overline{M}(\sqrt{\rho_n}) \right|^2 \sum_{\mathbf{p} \in \mathcal{W}_{\rho_n}} \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2 \leq \left| \overline{M}(\sqrt{\rho_n}) \right|^2 \|v\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.10)$$

□

Definition 4.2. Define a subspace of $L^2(\Omega)$ as follows

$$\mathcal{G}_\gamma(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{\mathbf{p} \in \mathbb{N}^d} \left| \overline{M}(|\mathbf{p}|) \right|^{2\gamma} e^{2T\overline{M}(|\mathbf{p}|)} \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2 < \infty \right\} \quad (4.11)$$

for any $\gamma \geq 0$. The norm of $v \in \mathcal{G}_\gamma(\Omega)$ is given by

$$\|v\|_{\mathcal{G}_\gamma(\Omega)} = \sqrt{\sum_{\mathbf{p} \in \mathbb{N}^d} \left| \overline{M}(|\mathbf{p}|) \right|^{2\gamma} e^{2T\overline{M}(|\mathbf{p}|)} \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2}. \quad (4.12)$$

Lemma 4.2. For $v \in \mathcal{G}_{1+\gamma}(\Omega)$, $\gamma \geq 0$

$$\|\mathbf{P}_{\rho_n} v - \mathbf{P}v\| \leq \left| \overline{M}(\sqrt{\rho_n}) \right|^{-\gamma} e^{-T\overline{M}(\sqrt{\rho_n})} \|v\|_{\mathcal{G}_{1+\gamma}(\Omega)}. \quad (4.13)$$

Proof. We have

$$\begin{aligned} \|\mathbf{P}_{\rho_n} v - \mathbf{P}v\|_{L^2(\Omega)}^2 &= \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_n}} \left| \overline{M}(|\mathbf{p}|) \right|^2 \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2 \\ &= \sum_{\mathbf{p} \notin \mathcal{W}_{\rho_n}} \left| \overline{M}(|\mathbf{p}|) \right|^{-2\gamma} e^{-2T\overline{M}(|\mathbf{p}|)} \left| \overline{M}(|\mathbf{p}|) \right|^{2+2\gamma} e^{2T\overline{M}(|\mathbf{p}|)} \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)}^2 \\ &\leq \left| \overline{M}(\sqrt{\rho_n}) \right|^{-2\gamma} e^{-2T\overline{M}(\sqrt{\rho_n})} \|v\|_{\mathcal{G}_{1+\gamma}(\Omega)}^2. \end{aligned}$$

□

4.1.1. The regularized solution and convergence rates. Since \mathbf{P} is an unbounded operator on $L^2(\Omega)$, we approximate it by the following operator \mathbf{P}_{ρ_n} defined above in equation (4.8). Since $\mathcal{A}(t)$ is an unbounded operator, we approximate it by a new approximate operator $\mathcal{A}(t) - \mathbf{P} + \mathbf{P}_{\rho_n}$. Moreover since F is a locally Lipschitz source function, we approximate F by F_Q defined by

$$F_Q(w(\mathbf{x}, t)) = \begin{cases} F(Q), & w(\mathbf{x}, t) > Q, \\ F(w(\mathbf{x}, t)), & -Q \leq w(\mathbf{x}, t) \leq Q, \\ F(-Q), & w(\mathbf{x}, t) < -Q. \end{cases} \quad (4.14)$$

for any $Q > 0$. In the sequel we use a parameter $Q_n := Q(n_1, n_2, \dots, n_d) \rightarrow +\infty$ as $|\mathbf{n}| \rightarrow +\infty$. So, when \mathbf{n} large enough, we have that $Q_n \geq \|\mathbf{u}\|_{L^\infty(0, T; L^2(\Omega))}$. Moreover, we also have

$$F_{Q_n}(\mathbf{u}(x, t)) = F(\mathbf{u}(x, t)), \quad \text{for } |\mathbf{n}| \text{ large enough.} \quad (4.15)$$

Using observation on p. 1250 in [40], we also obtain that F_{Q_n} is a globally Lipschitz source function in the following sense

$$\|F_{Q_n}(v_1) - F_{Q_n}(v_2)\|_{L^2(\Omega)} \leq 2K(Q_n) \|v_1 - v_2\|_{L^2(\Omega)}, \quad v_1, v_2 \in L^2(\Omega). \quad (4.16)$$

We consider a regularized problem below

$$\begin{cases} \frac{\partial \bar{U}_{\rho_n, \beta_n}}{\partial t} + \mathcal{A}(t) \bar{U}_{\rho_n, \beta_n} - \mathbf{P} \bar{U}_{\rho_n, \beta_n} + \mathbf{P}_{\rho_n} \bar{U}_{\rho_n, \beta_n} \\ = F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) + \hat{G}_{\beta_n}(\mathbf{x}, t), \quad 0 < t < T, \\ \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \\ \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, T) = \hat{H}_{\beta_n}(\mathbf{x}). \end{cases} \quad (4.17)$$

In the following Theorem, we obtain the existence, uniqueness and continuous dependence of the solutions for the proposed problem. We state the error estimation between the regularized solution and the exact solution. Our main result in this section is as follows

Theorem 4.1. *Let $H, G, H_{\beta_n}, G_{\beta_n}$ be as in Theorem 2.1. Let us choose β_n, ρ_n such that*

$$\lim_{|\mathbf{n}| \rightarrow +\infty} e^{2T\bar{M}(\sqrt{\rho_n})} \beta_n^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k} = \lim_{|\mathbf{n}| \rightarrow +\infty} e^{2T\bar{M}(\sqrt{\rho_n})} \beta_n^{-\mu_0} = 0. \quad (4.18)$$

Then Problem (4.17) has a unique solution $\bar{U}_{\rho_n, \beta_n} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega))$. Assume that Problem (4.1) has unique solution $\mathbf{u} \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; \mathcal{G}_{1+\gamma}(\Omega))$ for any $\gamma \geq 0$. Choose Q_n such that

$$\lim_{|\mathbf{n}| \rightarrow +\infty} e^{4K(Q_n)T} \bar{\Pi}(\mathbf{n}) = 0 \quad (4.19)$$

where

$$\bar{\Pi}(\mathbf{n}) = \max \left(e^{2T\bar{M}(\sqrt{\rho_n})} \beta_n^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_k}, e^{2T\bar{M}(\sqrt{\rho_n})} \beta_n^{-\mu_0}, \left| \bar{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} \right). \quad (4.20)$$

Then as $|\mathbf{n}| \rightarrow \infty$ the error

$$\mathbf{E} \left\| \bar{U}_{\rho_n, \beta_n}(\cdot, t) - u(\cdot, t) \right\|_{L^2(\Omega)}^2 \quad \text{is of order} \quad e^{\left(4K(Q_n)+2\right)(T-t)} e^{-2t\bar{M}(\sqrt{\rho_n})} \bar{\Pi}(\mathbf{n}). \quad (4.21)$$

Remark 4.1. *Thanks to Remark (3.1), we give one choice for β_n as follows*

$$\beta_n = \left(\prod_{k=1}^d n_k \right)^{\frac{1}{2\alpha_0 + d/2}} \quad (4.22)$$

where $0 < \alpha_0 < \frac{\mu_0}{2}$. Then we choose ρ_n such that

$$\bar{M}(\sqrt{\rho_n}) = \frac{\alpha_0}{T(2\alpha_0 + d/2)} \log \left(\prod_{k=1}^d n_k \right). \quad (4.23)$$

A simple computation gives that

$$\bar{\Pi}(\mathbf{n}) = \max \left(\frac{1}{\prod_{k=1}^d (n_k)^{4\mu_k - 1}}, \left(\prod_{k=1}^d n_k \right)^{\frac{2\alpha_0 - \mu_0}{2\alpha_0 + d/2}}, \left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 \delta}{4T\alpha_0 + dT}} \right). \quad (4.24)$$

Since

$$\lim_{|\mathbf{n}| \rightarrow +\infty} e^{4K(Q_n)T} \bar{\Pi}(\mathbf{n}) = 0 \quad (4.25)$$

we can take $K(Q_n)$ such that $e^{4K(Q_n)T} = (\bar{\Pi}(\mathbf{n}))^{\delta_0 - 1}$ for any $0 < \delta_0 < 1$. So, we have

$$K(Q_n) := \frac{\delta_0 - 1}{4T} \log(\bar{\Pi}(\mathbf{n})). \quad (4.26)$$

Since $\alpha_0 < \frac{\mu_0}{2}$ and $4\mu_k > 1$, $k = \overline{1, d}$, using (4.24), we deduce that $\lim_{|\mathbf{n}| \rightarrow \infty} \bar{\Pi}(\mathbf{n}) = 0$. Hence, we need to choose \mathbf{n} large enough such that $\bar{\Pi}(\mathbf{n}) < 1$. So, the equality (4.26) is suitable which leads to a chosen Q_n .

We state the two corollaries of the Theorem 4.1 next

Corollary 4.1. *Let us take two functions Γ_0, Γ_1 which are continuous functions on $[0, T]$. Assume that*

$$m_0 = \min \left(\min_{0 \leq t \leq T} |\Gamma_0(t)|, \min_{0 \leq t \leq T} |\Gamma_1(t)| \right) > 0 \quad (4.27)$$

and

$$m_1 = \max \left(\max_{0 \leq t \leq T} |\Gamma_0(t)|, \max_{0 \leq t \leq T} |\Gamma_1(t)| \right) > 0. \quad (4.28)$$

In Problem (4.1), let $\mathcal{A}(t)\mathbf{u} = -\Gamma_0(t)\Delta\mathbf{u} + \Gamma_1(t)\Delta^2\mathbf{u}$ and $F(\mathbf{u}) = \mathbf{u} - \mathbf{u}^3$. Then we get the backward in time problem for **extended Fisher-Kolmogorov equation** with time dependent coefficients as follows

$$\begin{cases} \mathbf{u}_t - \Gamma_0(t)\Delta\mathbf{u} + \Gamma_1(t)\Delta^2\mathbf{u} = \mathbf{u} - \mathbf{u}^3 + G(\mathbf{x}, t), \mathbf{x} \in \Omega, 0 < t < T, \\ \mathbf{u}(\mathbf{x}, t) = \Delta\mathbf{u}(\mathbf{x}, t) = 0, \mathbf{x} \in \partial\Omega, 0 \leq t \leq T, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), \mathbf{x} \in \Omega. \end{cases} \quad (4.29)$$

From Definition (4.1), let the operator \mathbf{P} be as follows

$$\mathbf{P}v := m_1 \sum_{\mathbf{p} \in \mathbb{N}^d} (\mathbf{p}^2 + \mathbf{p}^4) < v, \psi_{\mathbf{p}} > \psi_{\mathbf{p}}. \quad (4.30)$$

for any $v \in L^2(\Omega)$. It is easy to show that the pair of operators $(\mathcal{A}(t), \mathbf{P})$ as above satisfies Assumption (A) in Defintion (4.1). It is easy to see that the eigenvalues of \mathbf{P} are $\overline{M}(\mathbf{p}) = m_1(\mathbf{p}^2 + \mathbf{p}^4)$. Next, we find the operator \mathbf{P}_{ρ_n} by truncating Fourier series in (4.63) and we have

$$\mathbf{P}_{\rho_n} v := m_1 \sum_{|\mathbf{p}| \leq \sqrt{\frac{\rho_n}{m_1}}} (\mathbf{p}^2 + \mathbf{p}^4) < v, \psi_{\mathbf{p}} > \psi_{\mathbf{p}}(\mathbf{x}). \quad (4.31)$$

Thanks to (4.17), a regularized problem for (4.29) is given below

$$\begin{cases} \frac{\partial \overline{U}_{\rho_n, \beta_n}}{\partial t} - \Gamma_0(t)\Delta\overline{U}_{\rho_n, \beta_n} + \Gamma_1(t)\Delta^2\overline{U}_{\rho_n, \beta_n} - \mathbf{P}\overline{U}_{\rho_n, \beta_n} + \mathbf{P}_{\rho_n}\overline{U}_{\rho_n, \beta_n} \\ = F_{Q_n}(\overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) + \widehat{G}_{\beta_n}(\mathbf{x}, t), 0 < t < T, \\ \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) = 0, \mathbf{x} \in \partial\Omega, \\ \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, T) = \widehat{H}_{\beta_n}(\mathbf{x}), \end{cases} \quad (4.32)$$

where

$$F_{Q_n}(\overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) = \begin{cases} Q_n - Q_n^3, & \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) > Q_n, \\ \overline{U}_{\rho_n, \beta_n} - (\overline{U}_{\rho_n, \beta_n})^3, & -Q_n \leq \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) \leq Q_n, \\ -Q_n + Q_n^3, & \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) < -Q_n. \end{cases}$$

Assume that H, G, \mathbf{u} be as in Theorem (4.1). Let β_n be as in (4.22). Let ρ_n and Q_n be such that

$$\rho_n + \rho_n^2 = \frac{2\alpha_0}{m_1 T(4\alpha_0 + d)} \log \left(\prod_{k=1}^d n_k \right) \quad (4.33)$$

and

$$1 + 3Q_n^2 = \frac{\delta_0 - 1}{4T} \log(\overline{\Pi}(\mathbf{n})) \quad (4.34)$$

respectively. The last two equations can be solved by a simple way that leads to the value of ρ_n and Q_n . Then the error between the solution \mathbf{u} of Problem (4.29) and the solution $\overline{U}_{\rho_n, \beta_n}$ of Problem (4.32) is of order

$$\max \left[\frac{1}{\prod_{k=1}^d (n_k)^{4\mu_k \delta_0 - \delta_0}}, \left(\prod_{k=1}^d n_k \right)^{\frac{2\alpha_0 \delta_0 - \mu_0 \delta_0}{2\alpha_0 + d/2}}, \left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 \delta \delta_0}{4T\alpha_0 + dT}} \right] \left(\prod_{k=1}^d n_k \right)^{\frac{-2\alpha_0 t}{2T\alpha_0 + dT/2}}. \quad (4.35)$$

It is easy to check that the term in (4.35) tends to zero as $|\mathbf{n}| \rightarrow +\infty$.

4.1.2. Proof of the main results.

Proof of Theorem 4.1. Step 1. *The existence and uniqueness of the regularized problem.* We refer to the proof of Theorem 4.2 where we prove existence and uniqueness for the more general operator $\mathcal{A}(t, u)$ which is more general than the operator $\mathcal{A}(t)$ in Theorem 4.1 and $\mathcal{A}(t, u) = \mathcal{A}(t)$.

Step 2. *Regularity of the regularized solution $\bar{U}_{\rho_n, \beta_n}$.*

Let us define the function

$$\tilde{U}_{\rho_n, \beta_n}(\mathbf{x}, t) = e^{\kappa_n(t-T)} \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t), \quad (4.36)$$

where κ_n is positive constant to be selected later. By taking the partial derivative of $\tilde{U}_{\rho_n, \beta_n}(\mathbf{x}, t)$ with respect to t , we obtain

$$\begin{aligned} \frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial t}(\mathbf{x}, t) &= e^{\kappa_n(t-T)} \frac{\partial \bar{U}_{\rho_n, \beta_n}}{\partial t}(\mathbf{x}, t) + \kappa_n e^{\kappa_n(t-T)} \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) \\ &= -e^{\kappa_n(t-T)} \left(\mathcal{A}(t) - \mathbf{P} + \mathbf{P}_{\rho_n} \right) \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) + e^{\kappa_n(t-T)} F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) \\ &\quad + e^{\kappa_n(t-T)} \hat{G}_{\rho_n}(\mathbf{x}, t) + \kappa_n e^{\kappa_n(t-T)} \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) \\ &= - \left(\mathcal{A}(t) - \mathbf{P} + \mathbf{P}_{\rho_n} \right) \tilde{U}_{\rho_n, \beta_n}(\mathbf{x}, t) + \kappa_n \tilde{U}_{\rho_n, \beta_n}(\mathbf{x}, t) \\ &\quad + e^{\kappa_n(t-T)} \hat{G}_{\beta_n}(\mathbf{x}, t) + e^{\kappa_n(t-T)} F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)). \end{aligned}$$

Taking the inner product of both sides with $\tilde{U}_{\rho_n, \beta_n}(\mathbf{x}, t)$ gives

$$\begin{aligned} &\frac{1}{2} \frac{\partial}{\partial t} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &= \underbrace{\left\langle (\mathbf{P} - \mathcal{A}(t)) \tilde{U}_{\rho_n, \beta_n}(\cdot, t), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)}}_{=: \mathcal{J}_1} + \underbrace{\left\langle \mathbf{P}_{\rho_n} \tilde{U}_{\rho_n, \beta_n}(\cdot, t), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)}}_{=: \mathcal{J}_2} \\ &\quad + \kappa_n \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\quad + \underbrace{\left\langle e^{\kappa_n(t-T)} F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\cdot, t)), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)}}_{=: \mathcal{J}_3} + \underbrace{\left\langle e^{\kappa_n(t-T)} \hat{G}_{\beta_n}(\cdot, t), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)}}_{=: \mathcal{J}_4}. \end{aligned} \quad (4.37)$$

It remains to estimate $\mathcal{J}_1, \mathcal{J}_2$ and \mathcal{J}_3 . By the conditions for \mathbf{P} in Assumption (A) in Definition 4.1, we obtain that

$$\mathcal{J}_1 \geq \widehat{M} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{V(\Omega)}^2. \quad (4.38)$$

The term \mathcal{J}_2 can be estimated as follows

$$|\mathcal{J}_2| \leq \|\mathbf{P}_{\rho_n}\|_{\mathbb{L}(L^2(\Omega); L^2(\Omega))} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \leq \overline{M}(\sqrt{\rho_n}) \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2, \quad (4.39)$$

where we have used the inequality (4.10). Using Cauchy-Schwarz inequality and noting the globally Lipschitz property of the function F_{Q_n} , we deduce that

$$\begin{aligned} |\mathcal{J}_3| &= \left\langle e^{\kappa_n(t-T)} F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\cdot, t)), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)} \\ &\leq \frac{1}{2} e^{2\kappa_n(t-T)} \|F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\cdot, t))\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} e^{2\kappa_n(t-T)} \left(2K(Q_n) \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)} + \|F(0)\|_{L^2(\Omega)} \right)^2 + \frac{1}{2} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\leq \left(2K^2 Q_n + 1 \right) \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 + e^{2\kappa_n(t-T)} \|F(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.40)$$

where we note that we have used above the fact that $F_{Q_n}(0) = F(0)$. Using Cauchy-Schwartz inequality, we have the bound of $|\mathcal{J}_4|$ as follows

$$\begin{aligned} |\mathcal{J}_4| &= \left| \left\langle e^{\kappa_n(t-T)} \hat{G}_{\beta_n}(\cdot, t), \tilde{U}_{\rho_n, \beta_n}(\cdot, t) \right\rangle_{L^2(\Omega)} \right| \\ &\leq \frac{1}{2} e^{2\kappa_n(t-T)} \|\hat{G}_{\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.41)$$

Combining (4.38), (4.39), (4.40), (4.41) we have that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 &\geq \widehat{M} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{V(\Omega)}^2 \\ &\quad + \kappa_n \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 - \overline{M}(\sqrt{\rho_n}) \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\quad - \left(2K^2(Q_n) + 1\right) \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_H^2 + e^{2\kappa_n(t-T)} \|F(0)\|_{L^2(\Omega)}^2. \end{aligned}$$

Integrating the last inequality over $[t, T]$ yields

$$\begin{aligned} &\mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 + e^{2\kappa_n(t-T)} \mathbf{E} \|\hat{G}_{\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 + (T-t) \|F(0)\|_{L^2(\Omega)}^2 \\ &\geq \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 + 2\widehat{M} \mathbf{E} \left[\int_t^T \|\tilde{U}_{\rho_n, \beta_n}(\cdot, \tau)\|_{V(\Omega)}^2 d\tau \right] \\ &\quad + \int_t^T \left(2\kappa_n - 2\overline{M}(\sqrt{\rho_n}) - 4K^2(Q_n) - 2\right) \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned}$$

By choosing $\kappa_n = \overline{M}(\sqrt{\rho_n})$, we derive that

$$\begin{aligned} e^{2(t-T)\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\overline{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \left(4K^2(Q_n) + 2\right) \int_t^T e^{2(\tau-T)\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\quad + \mathbf{E} \|\hat{H}_{\beta_n}\|_{L^2(\Omega)}^2 + e^{2(t-T)\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{G}_{\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\quad + (T-t) \|F(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.42)$$

Multiplying both sides of the last inequality by $e^{2T\overline{M}(\sqrt{\rho_n})}$, we get

$$\begin{aligned} e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\overline{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \left(4K^2(Q_n) + 2\right) \int_t^T e^{2\tau\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\quad + e^{2T\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{H}_{\beta_n}\|_{L^2(\Omega)}^2 + e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{G}_{\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\quad + e^{2T\overline{M}(\sqrt{\rho_n})} (T-t) \|F(0)\|_{L^2(\Omega)}^2. \end{aligned}$$

Noting that

$$e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{G}_{\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \leq e^{2T\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{G}_{\beta_n}\|_{L^\infty(0, T; L^2(\Omega))}^2,$$

we deduce that

$$\begin{aligned} e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\overline{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 &\leq \left(4K^2(Q_n) + 2\right) \int_t^T e^{2\tau\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\tilde{U}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\quad + e^{2T\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{H}_{\beta_n}\|_{L^2(\Omega)}^2 + e^{2T\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\hat{G}_{\beta_n}\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ &\quad + e^{2T\overline{M}(\sqrt{\rho_n})} T \|F(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.43)$$

Applying Gronwall's inequality to the last inequality, we get

$$\begin{aligned} &e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \|\overline{U}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\leq \exp\left(4K^2(Q_n)(T-t)\right) e^{2T\overline{M}(\sqrt{\rho_n})} \left[\mathbf{E} \|\hat{H}_{\beta_n}\|_{L^2(\Omega)}^2 + \mathbf{E} \|\hat{G}_{\beta_n}\|_{L^\infty(0, T; L^2(\Omega))}^2 + T \|F(0)\|_{L^2(\Omega)}^2\right]. \end{aligned} \quad (4.44)$$

Multiplying both sides of the last inequality with $e^{-2t\overline{M}(\sqrt{\rho_n})}$, we get the upper bound of $\mathbf{E} \|\overline{U}_{\rho_n}(\cdot, t)\|_{L^2(\Omega)}^2$ which shows the stability of $\overline{U}_{\rho_n, \beta_n}(\cdot, t)$ in the sense of the solution $\overline{U}_{\rho_n, \beta_n}(\cdot, t)$ depend continuously on the given data \hat{H}_{β_n} , \hat{G}_{β_n} and F .

Step 3. *Error estimate between the regularized solution and the sought solution.*

It is easy to see that \mathbf{u} satisfies

$$\begin{aligned} \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \mathbf{P}_{\rho_n} \mathbf{u}(\mathbf{x}, t) &= F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t) \\ &\quad + (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}(\mathbf{x}, t) + (\mathbf{P} - \mathcal{A}(t)) \mathbf{u}(\mathbf{x}, t). \end{aligned}$$

Putting

$$\tilde{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) = \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t),$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) + \mathbf{P}_{\rho_n} \tilde{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) &= F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) - F(\mathbf{u}(\mathbf{x}, t)) - (\mathbf{P}_{\rho_n} - \mathbf{P})\mathbf{u}(\mathbf{x}, t) \\ &\quad + \left(\mathbf{P} - \mathcal{A}(t) \right) \tilde{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) + \hat{G}_{\beta_n}(\mathbf{x}, t) - G(\mathbf{x}, t). \end{aligned}$$

Put

$$\mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, t) = e^{\kappa_n(t-T)} \tilde{Z}_{\rho_n, \beta_n}(\mathbf{x}, t).$$

Take the inner product of the both sides of the last equality by $\mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, t)$ and then by integrating with respect to the time variable, it follows that

$$\begin{aligned} &\|\mathbf{X}_{\rho_n, \beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 - \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &= 2\kappa_n \int_t^T \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau - 2 \underbrace{\int_t^T \int_{\Omega} \left(\mathbf{P}_{\rho_n} \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) \right) \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau}_{:= \mathcal{J}_4} \\ &\quad + \underbrace{2e^{\kappa_n(t-T)} \int_t^T \int_{\Omega} F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, \tau)) - F(\mathbf{u}(\mathbf{x}, \tau)) \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau}_{:= \mathcal{J}_5} \\ &\quad + \underbrace{2e^{\kappa_n(t-T)} \int_t^T \int_{\Omega} \hat{G}_{\beta_n}(\mathbf{x}, \tau) - G(\mathbf{x}, \tau) \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau}_{:= \mathcal{J}_6} \\ &\quad + \underbrace{2e^{\kappa_n(t-T)} \int_t^T \int_{\Omega} (\mathbf{P} - \mathcal{A}(\tau)) \tilde{Z}_{\rho_n}(\mathbf{x}, \tau) \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau}_{:= \mathcal{J}_7} \\ &\quad + \underbrace{2e^{\kappa_n(t-T)} \int_t^T \int_{\Omega} (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}(\mathbf{x}, \tau) \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau}_{:= \mathcal{J}_8}. \end{aligned} \tag{4.45}$$

By the Cauchy-Schwartz inequality, the expectation of absolute of \mathcal{J}_4 is bounded by

$$\begin{aligned} \mathbf{E}|\mathcal{J}_4| &\leq 2\mathbf{E} \left[\int_t^T \sqrt{\left(\int_{\Omega} |\mathbf{P}_{\rho_n} \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau)|^2 d\mathbf{x} \right) \left(\int_{\Omega} |\mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau)|^2 d\mathbf{x} \right)} d\tau \right] \\ &\leq 2\bar{M}(\sqrt{\rho_n}) \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned} \tag{4.46}$$

For $|\mathbf{n}|$ large enough, we recall that $F_{Q_n}(\mathbf{u}) = F(\mathbf{u})$ and using the global Lipschitz property of F_{Q_n} , we have the bound of \mathcal{J}_5 as follows by using Cauchy-Schwartz inequality

$$\begin{aligned} \mathbf{E}|\mathcal{J}_5| &\leq 2\mathbf{E} \left[\int_t^T \left| F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, \tau)) - F(\mathbf{u}(\mathbf{x}, \tau)) \right|_{L^2(\Omega)} \left\| \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) \right\|^2 d\tau \right] \\ &= 2\mathbf{E} \left[\int_t^T \left\| F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, \tau)) - F_{Q_n}(\mathbf{u}(\mathbf{x}, \tau)) \right\|^2 \left\| \mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau) \right\|_{L^2(\Omega)} d\tau \right] \\ &\leq 4K(Q_n) \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned} \tag{4.47}$$

The term \mathcal{J}_6 is bounded by

$$\begin{aligned} \mathbf{E}|\mathcal{J}_6| &\leq \mathbf{E} \left[\int_t^T \left(\int_{\Omega} |\hat{G}_{\beta_n}(\mathbf{x}, \tau) - G(\mathbf{x}, \tau)|^2 d\mathbf{x} \right) d\tau \right] + \mathbf{E} \left[\int_t^T \left(\int_{\Omega} |\mathbf{X}_{\rho_n, \beta_n}(\mathbf{x}, \tau)|^2 d\mathbf{x} \right) d\tau \right] \\ &\leq T\mathbf{E} \left\| \hat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 + \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned} \tag{4.48}$$

The term \mathcal{J}_7 is estimated using the Assumption (A) in Definition 4.1 as follows

$$\begin{aligned}\mathbf{E}|\mathcal{J}_7| &= \mathbf{E} \left[\int_t^T \left\langle (\mathbf{P} - \mathcal{A}(\tau)) \mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau), \mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau) \right\rangle_{L^2(\Omega)} d\tau \right] \\ &\geq \widehat{M} \mathbf{E} \int_t^T \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{V(\Omega)}^2 d\tau\end{aligned}$$

and using Lemma 4.2

$$\begin{aligned}\mathbf{E}|\mathcal{J}_8| &= \mathbf{E} \left[\int_t^T 2e^{\kappa_n(\tau-T)} \int_{\Omega} (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}(\mathbf{x}, \tau) \mathbf{X}_{\rho_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau \right] \\ &\leq \left[\int_t^T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} e^{-2T\overline{M}(\sqrt{\rho_n})} \left\| \mathbf{u}(\cdot, \tau) \right\|_{\mathcal{G}_{1+\gamma}(\Omega)}^2 d\tau \right] + \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\leq T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} e^{-2T\overline{M}(\sqrt{\rho_n})} \left\| \mathbf{u} \right\|_{L^\infty(0, T; \mathcal{G}_{1+\gamma}(\Omega))}^2 + \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau.\end{aligned}\quad (4.49)$$

Combining (4.45), (4.46), (4.47), (4.48) (4.49), (4.49) gives

$$\begin{aligned}&\mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 - \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\geq \left(2\kappa_n - 2\overline{M}(\sqrt{\rho_n}) - 4K(Q_n) - 2 \right) \int_t^T \mathbf{E} \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &+ \widehat{M} \mathbf{E} \int_t^T \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{V(\Omega)}^2 d\tau - T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} e^{-2T\overline{M}(\sqrt{\rho_n})} \left\| \mathbf{u} \right\|_{L^\infty(0, T; \mathcal{G}_{1+\gamma}(\Omega))}^2 \\ &- T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0, T; L^2(\Omega))}^2.\end{aligned}\quad (4.50)$$

This leads to

$$\begin{aligned}&e^{2\kappa_n(t-T)} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 + \widehat{M} \mathbf{E} \int_t^T \|\mathbf{X}_{\rho_n, \beta_n}(\cdot, \tau)\|_{V(\Omega)}^2 d\tau \\ &+ \left(2\kappa_n - 2\overline{M}(\sqrt{\rho_n}) - 4K(Q_n) - 2 \right) \int_t^T e^{2\kappa_n(\tau-T)} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, \tau) - \mathbf{u}(\cdot, \tau) \right\|_{L^2(\Omega)}^2 d\tau \\ &\leq \mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ &+ T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} e^{-2T\overline{M}(\sqrt{\rho_n})} \left\| \mathbf{u} \right\|_{L^\infty(0, T; \mathcal{G}_{1+\gamma}(\Omega))}^2.\end{aligned}$$

Let us choose $\kappa_n = \overline{M}(\sqrt{\rho_n})$ and multiply both sides of the last inequality with $e^{2T\overline{M}(\sqrt{\rho_n})}$, then we conclude that

$$\begin{aligned}&e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 \\ &\leq e^{2T\overline{M}(\sqrt{\rho_n})} \left(\mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \right) \\ &+ T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} \left\| \mathbf{u} \right\|_{L^\infty(0, T; \mathcal{G}_{1+\gamma}(\Omega))}^2 \\ &+ \left(4K(Q_n) + 2 \right) \int_t^T e^{2\tau\overline{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, \tau) - \mathbf{u}(\cdot, \tau) \right\|_{L^2(\Omega)}^2 d\tau.\end{aligned}\quad (4.51)$$

The Gronwall's inequality implies that

$$\begin{aligned}&e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 \\ &\leq e^{\left(4K(Q_n) + 2 \right) (T-t)} e^{2T\overline{M}(\sqrt{\rho_n})} \left(\mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 + T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \right)\end{aligned}$$

$$+ e^{\left(4K(Q_n)+2\right)(T-t)} T \left| \overline{M}(|\sqrt{\rho_n}|) \right|^{-2\gamma} \left\| \mathbf{u} \right\|_{L^\infty(0,T;\mathcal{G}_{1+\gamma}(\Omega))}^2. \quad (4.52)$$

Multiplying both sides of (4.52) with $e^{-2t\overline{M}(\sqrt{\rho_n})}$ and thanks to Corollary (2.1), we conclude that (4.21) holds. \square

4.2. General problem with coefficients that depend on t and \mathbf{u} . Let $H \in L^2(\Omega)$. In this section, we consider the problem of constructing an $\mathbf{u} \in C([0, T; V(\Omega)])$, $\mathbf{u}' \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{cases} \mathbf{u}_t + \mathcal{A}(t, \mathbf{u})\mathbf{u} = F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t), & \mathbf{x} \in \Omega, 0 < t < T, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \quad (4.53)$$

Let $0 < R < \infty$. Define the following set

$$B_R(L^2(\Omega)) := \left\{ w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq R \right\}. \quad (4.54)$$

Definition 4.3. The pair of operators $(\mathcal{A}(t, w), \mathbf{P})$ satisfies Assumption (B) if the following conditions hold

(a) For any $v \in L^2(\Omega)$, there exists a increasing function \overline{M} such as

$$\mathbf{P}v(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{N}^d} \overline{M}(|\mathbf{p}|) \langle v, \psi_{\mathbf{p}} \rangle_{L^2(\Omega)} \psi_{\mathbf{p}}(\mathbf{x}). \quad (4.55)$$

(b) For $w \in B_R(L^2(\Omega))$, $u \in \mathcal{V}(\Omega)$, $v \in \mathcal{V}(\Omega)$ then

$$\left\langle (\mathbf{P} - \mathcal{A}(t, w))u, v \right\rangle_{L^2(\Omega)} \leq \widetilde{M}_a \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)}. \quad (4.56)$$

(c) For $w \in B_R(L^2(\Omega))$ and any $v \in \mathcal{V}(\Omega)$, there exists $\widehat{M} > 0$ such that

$$\left\langle (\mathbf{P} - \mathcal{A}(t, w))v, v \right\rangle_{L^2(\Omega)} \geq \widehat{M} \|v\|_{\mathcal{V}(\Omega)}^2. \quad (4.57)$$

(d) There exists $\widetilde{M} > 0$ such that

$$\left| \left\langle (\mathcal{A}(t, w_1) - \mathcal{A}(t, w_2))u, v \right\rangle_{L^2(\Omega)} \right| \leq \widetilde{M} \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)} \|w_1 - w_2\|_{L^2(\Omega)}, \quad (4.58)$$

for any $u \in \mathcal{V}(\Omega)$, $v \in \mathcal{V}(\Omega)$ and $w_1, w_2 \in B_R(L^2(\Omega))$.

Remark 4.2. We give some particular equations of the model (4.53). Let us choose $\mathcal{A}(t, \mathbf{u}) = \nabla(D(\mathbf{u})\nabla\mathbf{u})$ in the first equation of (4.53) then this equation is called a logistic reaction-diffusion equation. Here \mathbf{u} represents the population density of species at location x and time t , $D(\mathbf{u})$ is the density dependent diffusion coefficient, the notation ∇ is the usual gradient operator and $F(\mathbf{u})$ is a logistic type source term.

- When $F(\mathbf{u}) = a\mathbf{u}(1 - \mathbf{u})$ $a > 0$, Problem (4.53) is called backward in time for **Fisher-type logistic equations**. (See [28]).
- When $F(\mathbf{u}) = a\mathbf{u}^2(1 - \mathbf{u})$ $a > 0$, Problem (4.53) is called backward in time for **Huxley equation**.
- When $F(\mathbf{u}) = a\mathbf{u}^2(1 - \mathbf{u})(\mathbf{u} - \theta_1)$ $a > 0$, Problem (4.53) is called backward in time for **Fitzhugh-Nagumo equation**.

Some more applications in biology of the above equations and generalized problem can be found in [7].

Using a similar method as in previous subsection, we present a regularized problem for Problem (4.53) as follows

$$\begin{cases} \frac{\partial \overline{U}_{\rho_n, \beta_n}}{\partial t} + \mathcal{A}(t, \overline{U}_{\rho_n, \beta_n})\overline{U}_{\rho_n, \beta_n} - \mathbf{P}\overline{U}_{\rho_n, \beta_n} + \mathbf{P}_{\rho_n}\overline{U}_{\rho_n} = \\ F_{Q_n}(\overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) + \widehat{G}_{\beta_n}(\mathbf{x}, t), & 0 < t < T, \\ \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, t) = 0, & x \in \partial\Omega, \\ \overline{U}_{\rho_n, \beta_n}(\mathbf{x}, T) = \widehat{H}_{\beta_n}(\mathbf{x}). \end{cases} \quad (4.59)$$

Theorem 4.2. Let $H, G, \mathbf{u}, \beta_{\mathbf{n}}, \rho_{\mathbf{n}}$ be as Theorem 4.1. Then the system (4.59) has a unique solution $\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega))$. Choose $Q_{\mathbf{n}}$ as in Theorem 4.1. Then for \mathbf{n} large enough, the error

$$\mathbb{E} \left\| \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 \quad \text{is of order} \quad e^{\left(4K(Q_{\mathbf{n}})+2+\frac{4\widetilde{M}M_0}{M}\right)(T-t)} e^{-2t\overline{M}(\sqrt{\rho_{\mathbf{n}}})\overline{\Pi}(\mathbf{n})}. \quad (4.60)$$

Where the term $\overline{\Pi}(\mathbf{n})$ above is defined in equation (4.20).

Remark 4.3. One example for choices of $\beta_{\mathbf{n}}, \rho_{\mathbf{n}}, Q_{\mathbf{n}}$ are given in Remark (4.1).

Corollary 4.2. Consider the following problem for **Huxley equation**

$$\begin{cases} \mathbf{u}_t - \nabla \left(D(\mathbf{u}) \nabla \mathbf{u} \right) = \mathbf{u}^2(1 - \mathbf{u}) + G(\mathbf{x}, t), & \mathbf{x} \in \Omega, 0 < t < T, \\ \mathbf{u}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \mathbf{u}(\mathbf{x}, T) = H(\mathbf{x}), & \mathbf{x} \in \Omega \end{cases} \quad (4.61)$$

where the density dependent diffusion coefficient D satisfies that $D_0 \leq D(w(x, t)) \leq D_1$ for any $w \in L^2(\Omega)$ and D_0, D_1 are positive numbers. We assume that D is a globally Lipschitz function, i.e., there exists $\widetilde{M} \geq 0$ such that

$$\|D(w_1) - D(w_2)\|_{L^2(\Omega)} \leq \widetilde{M} \|w_1 - w_2\|_{L^2(\Omega)}. \quad (4.62)$$

From Definition (4.3), the operator \mathbf{P} can be chosen as follows

$$\mathbf{P}v := D_1 \Delta v = D_1 \sum_{\mathbf{p} \in \mathbb{N}^d} \mathbf{p}^2 \langle v, \psi_{\mathbf{p}} \rangle \psi_{\mathbf{p}}, \quad (4.63)$$

for any $v \in L^2(\Omega)$. It is easy to show that $(\mathcal{A}(t, w), \mathbf{P})$ as above, satisfies Assumption (B) in the sense of Definition (4.3). We can easily see that the eigenvalues of \mathbf{P} are $\overline{M}(\mathbf{p}) = D_1 \mathbf{p}^2$. Next, we find the operator $\mathbf{P}_{\rho_{\mathbf{n}}}$ by truncating Fourier series in (4.63) and we have

$$\mathbf{P}_{\rho_{\mathbf{n}}} v := m_1 \sum_{|\mathbf{p}| \leq \sqrt{\frac{\rho_{\mathbf{n}}}{D_1}}} \mathbf{p}^2 \langle v, \psi_{\mathbf{p}} \rangle \psi_{\mathbf{p}}. \quad (4.64)$$

Thanks to (4.59), a regularized problem for (4.61) is given below

$$\begin{cases} \frac{\partial \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}}{\partial t} - \nabla \left(D(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}) \nabla \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} \right) - \mathbf{P} \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} + \mathbf{P}_{\rho_{\mathbf{n}}} \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}} \\ = F_{Q_{\mathbf{n}}}(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t)) + \widehat{G}_{\beta_{\mathbf{n}}}(\mathbf{x}, t), & 0 < t < T, \\ \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega, \\ \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, T) = \widehat{H}_{\beta_{\mathbf{n}}}(\mathbf{x}), \end{cases} \quad (4.65)$$

where

$$F_{Q_{\mathbf{n}}}(\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t)) = \begin{cases} Q_{\mathbf{n}}^2 - Q_{\mathbf{n}}^3, & \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) > Q_{\mathbf{n}}, \\ (\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}})^2 - (\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}})^3, & -Q_{\mathbf{n}} \leq \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) \leq Q_{\mathbf{n}}, \\ -Q_{\mathbf{n}}^2 + Q_{\mathbf{n}}^3, & \overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}(\mathbf{x}, t) < -Q_{\mathbf{n}}. \end{cases}$$

Assume that H, G, \mathbf{u} are as in Theorem (4.1). Let $\beta_{\mathbf{n}}$ be as in (4.22). Let $\rho_{\mathbf{n}}$ and $Q_{\mathbf{n}}$ be such that

$$\rho_{\mathbf{n}} = \sqrt{\frac{2\alpha_0}{m_1 T(4\alpha_0 + d)} \log \left(\prod_{k=1}^d n_k \right)}. \quad (4.66)$$

and

$$2Q_{\mathbf{n}} + 3Q_{\mathbf{n}}^2 = \frac{\delta_0 - 1}{4T} \log(\overline{\Pi}(\mathbf{n})) \quad (4.67)$$

respectively. Then the error between the solution \mathbf{u} of Problem (4.61) and the solution $\overline{U}_{\rho_{\mathbf{n}}, \beta_{\mathbf{n}}}$ of Problem (4.65) is of order

$$\max \left[\frac{1}{\prod_{k=1}^d (n_k)^{4\mu_k \delta_0 - \delta_0}}, \left(\prod_{k=1}^d n_k \right)^{\frac{2\alpha_0 \delta_0 - \mu_0 \delta_0}{2\alpha_0 + d/2}}, \left(\prod_{k=1}^d n_k \right)^{\frac{-4\alpha_0 \delta \delta_0}{4T\alpha_0 + dT}} \right] \left(\prod_{k=1}^d n_k \right)^{\frac{-2\alpha_0 t}{2T\alpha_0 + dT/2}}. \quad (4.68)$$

It is easy to check that the term in (4.68) tends to zero as $|\mathbf{n}| \rightarrow +\infty$.

Proof of Theorem 4.2. The proof is divided into some steps.

Step 1. *The existence and uniqueness of the solution to problem (4.59).* Let $\bar{V}_{\rho_n, \beta_n}(\mathbf{x}, t) = \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, T - t)$ and define the following operator $\mathcal{B}(t, w) = \mathbf{P} - \mathcal{A}(t, w)$. Then it is obvious that $\bar{V}_{\rho_n, \beta_n}(\mathbf{x}, t)$ satisfies the following equation

$$\begin{cases} \frac{\partial \bar{V}_{\rho_n, \beta_n}}{\partial t} + \mathcal{B}(t, \bar{V}_{\rho_n, \beta_n}) \bar{V}_{\rho_n, \beta_n} - \mathbf{P} \bar{U}_{\rho_n, \beta_n} \\ = \mathbf{P}_{\rho_n} \bar{V}_{\rho_n, \beta_n} - F_{Q_n}(\bar{V}_{\rho_n, \beta_n}(\mathbf{x}, t)) - \hat{G}_{\beta_n}(\mathbf{x}, t), \quad 0 < t < T, \\ \bar{V}_{\rho_n, \beta_n}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \\ \bar{V}_{\rho_n, \beta_n}(\mathbf{x}, 0) = \hat{H}_{\beta_n}(\mathbf{x}). \end{cases} \quad (4.69)$$

By the assumptions on \mathcal{A} above in (4.55), (4.56), (4.57), (4.58), it is easy to show that

(i) For $w \in B_R(L^2(\Omega))$, $u \in \mathcal{V}(\Omega)$, $v \in \mathcal{V}(\Omega)$ then

$$\left\langle (\mathcal{B}(t, w)u, v) \right\rangle_{L^2(\Omega)} \leq \widetilde{M}_a \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)}. \quad (4.70)$$

(ii) For any $v \in \mathcal{V}(\Omega)$ then

$$\left\langle (\mathcal{B}(t, w)v, v) \right\rangle_{L^2(\Omega)} \geq \widehat{M} \|v\|_{\mathcal{V}(\Omega)}^2. \quad (4.71)$$

(iii) There exist a subspace $\mathcal{V}_1(\Omega) \subset L^2(\Omega)$ in which

$$\left| \left\langle (\mathcal{B}(t, w_1) - \mathcal{B}(t, w_2))u, v \right\rangle \right|_{L^2(\Omega)} \leq \widetilde{M} \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)} \|w_1 - w_2\|_{L^2(\Omega)}, \quad (4.72)$$

for any $u \in \mathcal{V}(\Omega)$, $v \in \mathcal{V}(\Omega)$ and $w_1, w_2 \in B_R(L^2(\Omega))$.

Hence, the conditions proved above for the operator \mathcal{B} show that \mathcal{B} satisfies the assumptions of Theorem 5.10 in [42] (page 252). With the help of Theorem 5.10 in of [42], we conclude that the Problem (4.69) has unique solution $\bar{V}_{\rho_n, \beta_n} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega))$.

Step 2. *The error estimate for $\mathbf{E} \left\| \bar{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}$.*

First, we have

$$\begin{aligned} \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \mathbf{P}_{\rho_n} \mathbf{u}(\mathbf{x}, t) &= F(\mathbf{u}(\mathbf{x}, t)) + G(\mathbf{x}, t) \\ &+ (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}(\mathbf{x}, t) + (\mathbf{P} - \mathcal{A}(t, \mathbf{u})) \mathbf{u}(\mathbf{x}, t), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \bar{U}_{\rho_n, \beta_n}}{\partial t} + \mathbf{P}_{\rho_n} \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) &= F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) + \hat{G}_{\beta_n}(\mathbf{x}, t) \\ &+ (\mathbf{P}_{\rho_n} - \mathbf{P}) \bar{U}_{\rho_n, \beta_n} + (\mathbf{P} - \mathcal{A}(t, \bar{U}_{\rho_n, \beta_n})) \bar{U}_{\rho_n, \beta_n}. \end{aligned} \quad (4.73)$$

Putting $\bar{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) = \bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t)$, we have

$$\begin{aligned} \frac{\partial \bar{Z}_{\rho_n, \beta_n}}{\partial t}(\mathbf{x}, t) + \mathbf{P}_{\rho_n} \bar{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) &= F_{Q_n}(\bar{U}_{\rho_n, \beta_n}(\mathbf{x}, t)) - F(\mathbf{u}(\mathbf{x}, t)) - (\mathbf{P}_{\rho_n} - \mathbf{P}) \mathbf{u}(\mathbf{x}, t) \\ &+ (\mathbf{P} - \mathcal{A}(t, \bar{U}_{\rho_n, \beta_n})) \bar{Z}_{\rho_n, \beta_n}(\mathbf{x}, t) + \hat{G}_{\beta_n}(\mathbf{x}, t) - G(\mathbf{x}, t) \\ &+ (\mathcal{A}(t, \mathbf{u}) - \mathcal{A}(t, \bar{U}_{\rho_n, \beta_n})) \mathbf{u}(\mathbf{x}, t). \end{aligned}$$

Put $\bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, t) = e^{\kappa_n(t-T)} \bar{Z}_{\rho_n, \beta_n}(\mathbf{x}, t)$, and taking the inner product of the last equality by $\bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, t)$, and integrating over (t, T) we have

$$\begin{aligned} &\|\bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, T)\|_{L^2(\Omega)}^2 - \|\bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, t)\|_{L^2(\Omega)}^2 \\ &= 2\kappa_n \int_t^T \|\bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, \tau)\|_{L^2(\Omega)}^2 d\tau - 2 \underbrace{\int_t^T \int_{\Omega} \mathbf{P}_{\rho_n, \beta_n} \bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, \tau) \bar{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, \tau) dx d\tau}_{:= \mathcal{I}_{4,4}} \end{aligned}$$

$$\begin{aligned}
& + \underbrace{2e^{\kappa_{\mathbf{n}}(t-T)} \int_t^T \int_{\Omega} F_{Q_{\mathbf{n}}}(\bar{U}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)) - F(\mathbf{u}(\mathbf{x},\tau)) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} d\tau}_{:=\mathcal{J}_{5,5}} \\
& + \underbrace{2e^{\kappa_{\mathbf{n}}(t-T)} \int_t^T \int_{\Omega} \hat{G}_{\beta_{\mathbf{n}}}(\mathbf{x},\tau) - G(\mathbf{x},\tau) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} d\tau}_{:=\mathcal{J}_{6,6}} \\
& + \underbrace{2e^{\kappa_{\mathbf{n}}(t-T)} \int_t^T \int_{\Omega} (\mathbf{P} - \mathcal{A}(t, \bar{U}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}})) \bar{Z}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} d\tau}_{:=\mathcal{J}_{7,7}} \\
& + \underbrace{2e^{\kappa_{\mathbf{n}}(t-T)} \int_t^T \int_{\Omega} (\mathbf{P}_{\rho_{\mathbf{n}}} - \mathbf{P}) u(\mathbf{x},\tau) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} d\tau}_{:=\mathcal{J}_{8,8}} \\
& + \underbrace{2 \int_t^T \int_{\Omega} e^{\kappa_{\mathbf{n}}(\tau-T)} (\mathcal{A}(\tau, \mathbf{u}) - \mathcal{A}(\tau, \bar{U}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}})) \mathbf{u}(\mathbf{x},\tau) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} d\tau}_{:=\mathcal{J}_{9,9}}. \tag{4.74}
\end{aligned}$$

By a similar techniques as in equation (4.46), we obtain

$$\mathbf{E}(\mathcal{J}_{4,4}) \geq -2\bar{M}(\sqrt{\rho_{\mathbf{n}}}) \int_t^T \mathbf{E} \|\bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \tag{4.75}$$

By a similar techniques as in equation (4.47), we obtain

$$\mathbf{E}(\mathcal{J}_{5,5}) \geq -4K(Q_{\mathbf{n}}) \int_t^T \mathbf{E} \|\bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \tag{4.76}$$

By a similar techniques as in equation (4.48), we obtain

$$\mathbf{E}(\mathcal{J}_{6,6}) \geq -T \mathbf{E} \|\hat{G}_{\beta_{\mathbf{n}}}(\cdot) - G(\cdot)\|_{L^\infty(0,T;L^2(\Omega))}^2 - \int_t^T \mathbf{E} \|\bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \tag{4.77}$$

By a similar techniques as in equation (4.49), we obtain

$$\begin{aligned}
\mathbf{E}(\mathcal{J}_{7,7}) &= \mathbf{E} \left[\int_t^T \left\langle (\mathbf{P} - \mathcal{A}(\tau, \bar{U}(\rho_{\mathbf{n}}, \beta_{\mathbf{n}}))) \mathbf{X}_{\rho_{\mathbf{n}}}(\cdot, \tau), \mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau) \right\rangle_{L^2(\Omega)} d\tau \right] \\
&\geq \widehat{M} \mathbf{E} \int_t^T \|\bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau)\|_{\mathcal{V}(\Omega)}^2 d\tau. \tag{4.78}
\end{aligned}$$

By a similar techniques as in equation (4.49), we obtain

$$\mathbf{E}(\mathcal{J}_{8,8}) \geq -T e^{-2T\bar{M}(\sqrt{\rho_{\mathbf{n}}})} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{G}_{1+\gamma}(\Omega))}^2 - \int_t^T \mathbf{E} \|\bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \tag{4.79}$$

Now, we turn to estimate $J_{9,9}$. For $\tau \in (0, T)$, using (4.58), we get

$$\begin{aligned}
& \int_{\Omega} e^{\kappa_{\mathbf{n}}(\tau-T)} (\mathcal{A}(\tau, \mathbf{u}) - \mathcal{A}(\tau, \bar{U}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}})) \mathbf{u}(\mathbf{x},\tau) \bar{\mathbf{X}}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau) d\mathbf{x} \\
& \leq \widetilde{M} \|\mathbf{u}(\cdot, \tau)\|_{\mathcal{V}(\Omega)} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{\mathcal{V}(\Omega)} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{L^2(\Omega)} \\
& \leq \widetilde{M} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V}(\Omega))} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{\mathcal{V}(\Omega)} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{L^2(\Omega)} \\
& \leq \widetilde{M} \widetilde{M}_0 \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{\mathcal{V}(\Omega)} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{L^2(\Omega)} \\
& \leq \widetilde{M} \widetilde{M}_0 \left(\frac{\widehat{M} \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{\mathcal{V}(\Omega)}^2}{2\widetilde{M}\widetilde{M}_0} + \frac{2\widetilde{M}\widetilde{M}_0 \|\mathbf{X}_{\rho_{\mathbf{n}},\beta_{\mathbf{n}}}(\mathbf{x},\tau)\|_{L^2(\Omega)}}{\widehat{M}} \right)
\end{aligned}$$

where \widetilde{M}_0 is a positive constant such that $\widetilde{M}_0 \geq \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{V}(\Omega))}$. This implies that

$$\begin{aligned} \mathbf{E} \Big|_{\mathcal{J}_{9,9}} &= \mathbf{E} \left(2 \int_t^T \int_\Omega e^{\kappa_n(\tau-T)} \left(\mathcal{A}(\tau, \mathbf{u}) - \mathcal{A}(\tau, \overline{U}_{\rho_n, \beta_n}) \right) \mathbf{u}(\mathbf{x}, \tau) \overline{\mathbf{X}}_{\rho_n, \beta_n}(\mathbf{x}, \tau) d\mathbf{x} d\tau \right) \\ &\leq \widehat{M} \mathbf{E} \left(\int_t^T \|\overline{\mathbf{X}}_{\rho_n, \beta_n}(\cdot, \tau)\|_{\mathcal{V}(\Omega)}^2 d\tau \right) + \frac{4\widetilde{M}\widetilde{M}_0}{\widehat{M}} \mathbf{E} \left(\int_t^T \|\overline{\mathbf{X}}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \right). \end{aligned} \quad (4.80)$$

Combining (4.74), (4.75), (4.76), (4.77), (4.78), (4.79), (4.80) gives

$$\begin{aligned} &\mathbf{E} \|\overline{\mathbf{X}}_{\rho_n, \beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 - \mathbf{E} \|\overline{\mathbf{X}}_{\rho_n, \beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\geq \left(2\kappa_n - 2\overline{M}(\sqrt{\rho_n}) - 4K(Q_n) - 2 - \frac{4\widetilde{M}\widetilde{M}_0}{\widehat{M}} \right) \int_t^T \mathbf{E} \|\overline{\mathbf{X}}_{\rho_n, \beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\ &\quad - T e^{-2T\overline{M}(\sqrt{\rho_n})} \|\mathbf{u}\|_{L^\infty(0,T;\mathcal{G}_{1+\gamma}(\Omega))}^2 - T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0,T;L^2(\Omega))}^2. \end{aligned}$$

Letting $\kappa_n = \overline{M}(\sqrt{\rho_n})$ and by using Gronwall's inequality as in the proof of Theorem 3.2, we conclude that

$$\begin{aligned} &e^{2t\overline{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \overline{U}_{\rho_n, \beta_n}(\cdot, t) - \mathbf{u}(\cdot, t) \right\|_{L^2(\Omega)}^2 \\ &\leq e^{\left(4K(Q_n) + 2 + \frac{4\widetilde{M}\widetilde{M}_0}{\widehat{M}} \right)(T-t)} e^{2T\overline{M}(\sqrt{\rho_n})} \mathbf{E} \left\| \widehat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 \\ &\quad + e^{\left(4K(Q_n) + 2 + \frac{4\widetilde{M}\widetilde{M}_0}{\widehat{M}} \right)(T-t)} e^{2T\overline{M}(\sqrt{\rho_n})} T \mathbf{E} \left\| \widehat{G}_{\beta_n}(\cdot) - G(\cdot) \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ &\quad + e^{\left(4K(Q_n) + 2 + \frac{4\widetilde{M}\widetilde{M}_0}{\widehat{M}} \right)(T-t)} T \left\| \mathbf{u} \right\|_{L^\infty(0,T;\mathcal{G}_{1+\gamma}(\Omega))}^2. \end{aligned} \quad (4.81)$$

Multiplying both sides of the last inequality with $e^{-2t\overline{M}(\sqrt{\rho_n})}$ and combining with the results in Corollary (2.1), we get the desired result (4.60). \square

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